15 | OSCILLATIONS

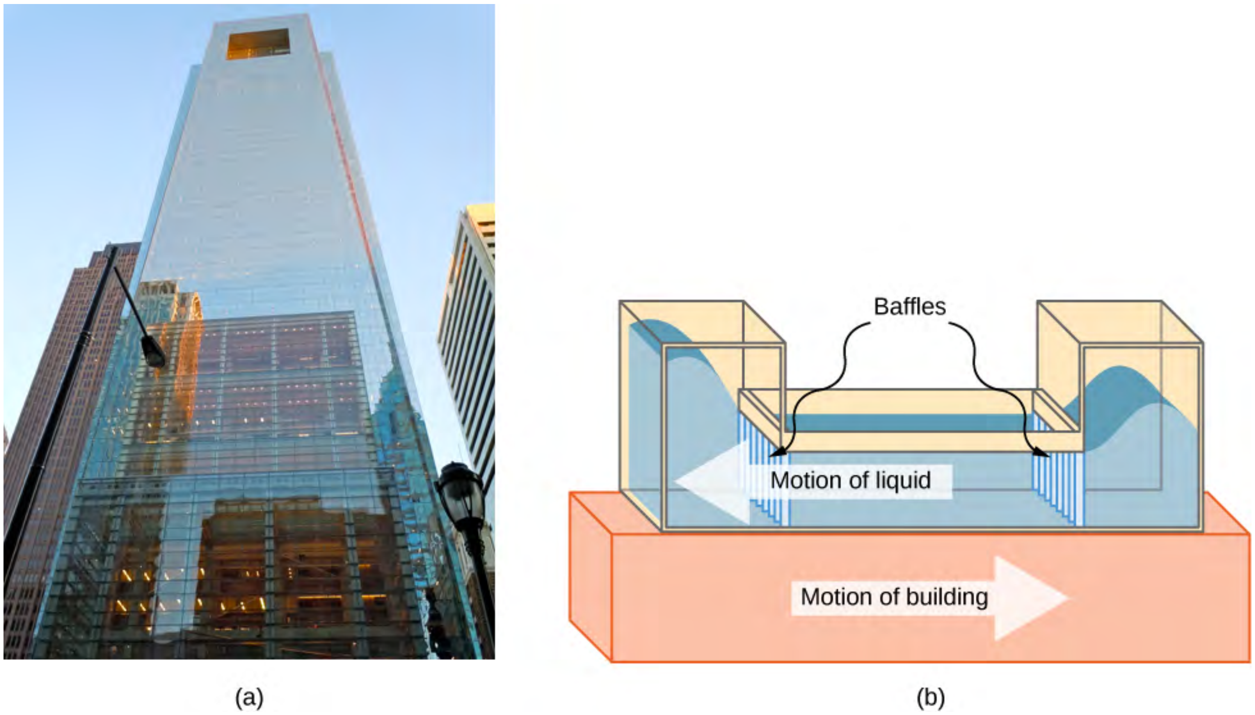


Figure 15.1 (a) The Comcast Building in Philadelphia, Pennsylvania, looming high above the skyline, is approximately 305 meters (1000 feet) tall. At this height, the top floors can oscillate back and forth due to seismic activity and fluctuating winds. (b) Shown above is a schematic drawing of a tuned, liquid-column mass damper, installed at the top of the Comcast, consisting of a 300,000-gallon reservoir of water to reduce oscillations.

Chapter Outline

Introduction

We begin the study of oscillations with simple systems of pendulums and springs. Although these systems may seem quite basic, the concepts involved have many real-life applications. For example, the Comcast Building in Philadelphia, Pennsylvania, stands approximately 305 meters (1000 feet) tall. As buildings are built taller, they can act as inverted, physical pendulums, with the top floors oscillating due to seismic activity and fluctuating winds. In the Comcast Building, a tuned-mass damper is used to reduce the oscillations. Installed at the top of the building is a tuned, liquid-column mass damper, consisting of a 300,000-gallon reservoir of water. This U-shaped tank allows the water to oscillate freely at a frequency that matches the natural frequency of the building. Damping is provided by tuning the turbulence levels in the moving water using baffles.

15.1 | Simple Harmonic Motion

Learning Objectives

By the end of this section, you will be able to:

- Define the terms period and frequency
- List the characteristics of simple harmonic motion
- Explain the concept of phase shift
- Write the equations of motion for the system of a mass and spring undergoing simple harmonic motion
- Describe the motion of a mass oscillating on a vertical spring

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time (**Figure 15.2**). The string vibrates around an equilibrium position, and one oscillation is completed when the string starts from the initial position, travels to one of the extreme positions, then to the other extreme position, and returns to its initial position. We define **periodic motion** to be any motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by a child swinging on a swing. In this section, we study the basic characteristics of oscillations and their mathematical description.

Figure 15.2 When a guitar string is plucked, the string oscillates up and down in periodic motion. The vibrating string causes the surrounding air molecules to oscillate, producing sound waves. (credit: Yutaka Tsutano)

Period and Frequency in Oscillations

In the absence of friction, the time to complete one oscillation remains constant and is called the **period (*T*)**. Its units are usually seconds, but may be any convenient unit of time. The word ‘period’ refers to the time for some event whether repetitive or not, but in this chapter, we shall deal primarily in periodic motion, which is by definition repetitive.

A concept closely related to period is the frequency of an event. **Frequency (*f*)** is defined to be the number of events per unit time. For periodic motion, frequency is the number of oscillations per unit time. The relationship between frequency and period is

$$f = \frac{1}{T}. \tag{15.1}$$

The SI unit for frequency is the *hertz* (Hz) and is defined as one *cycle per second*:

$$1\text{ Hz} = 1\frac{\text{cycle}}{\text{sec}} \text{ or } 1\text{ Hz} = \frac{1}{\text{s}} = 1\text{ s}^{-1}.$$

A cycle is one complete **oscillation**.

Example 15.1

Determining the Frequency of Medical Ultrasound

Ultrasound machines are used by medical professionals to make images for examining internal organs of the body. An ultrasound machine emits high-frequency sound waves, which reflect off the organs, and a computer receives the waves, using them to create a picture. We can use the formulas presented in this module to determine the frequency, based on what we know about oscillations. Consider a medical imaging device that produces ultrasound by oscillating with a period of $0.400\ \mu\text{s}$. What is the frequency of this oscillation?

Strategy

The period (*T*) is given and we are asked to find frequency (*f*).

Solution

Substitute $0.400\ \mu\text{s}$ for *T* in $f = \frac{1}{T}$:

$$f = \frac{1}{T} = \frac{1}{0.400 \times 10^{-6}\ \text{s}}.$$

Solve to find

$$f = 2.50 \times 10^{6}\ \text{Hz}.$$

Significance

This frequency of sound is much higher than the highest frequency that humans can hear (the range of human hearing is 20 Hz to 20,000 Hz); therefore, it is called ultrasound. Appropriate oscillations at this frequency generate ultrasound used for noninvasive medical diagnoses, such as observations of a fetus in the womb.

Characteristics of Simple Harmonic Motion

A very common type of periodic motion is called **simple harmonic motion (SHM)**. A system that oscillates with SHM is called a **simple harmonic oscillator**.

Simple Harmonic Motion

In simple harmonic motion, the acceleration of the system, and therefore the net force, is proportional to the displacement and acts in the opposite direction of the displacement.

A good example of SHM is an object with mass *m* attached to a spring on a frictionless surface, as shown in **Figure 15.3**. The object oscillates around the equilibrium position, and the net force on the object is equal to the force provided by the spring. This force obeys Hooke's law $F_s = -kx,$ as discussed in a previous chapter.

If the net force can be described by Hooke's law and there is no *damping* (slowing down due to friction or other nonconservative forces), then a simple harmonic oscillator oscillates with equal displacement on either side of the equilibrium position, as shown for an object on a spring in **Figure 15.3**. The maximum displacement from equilibrium is called the **amplitude (*A*)**. The units for amplitude and displacement are the same but depend on the type of oscillation. For the object on the spring, the units of amplitude and displacement are meters.

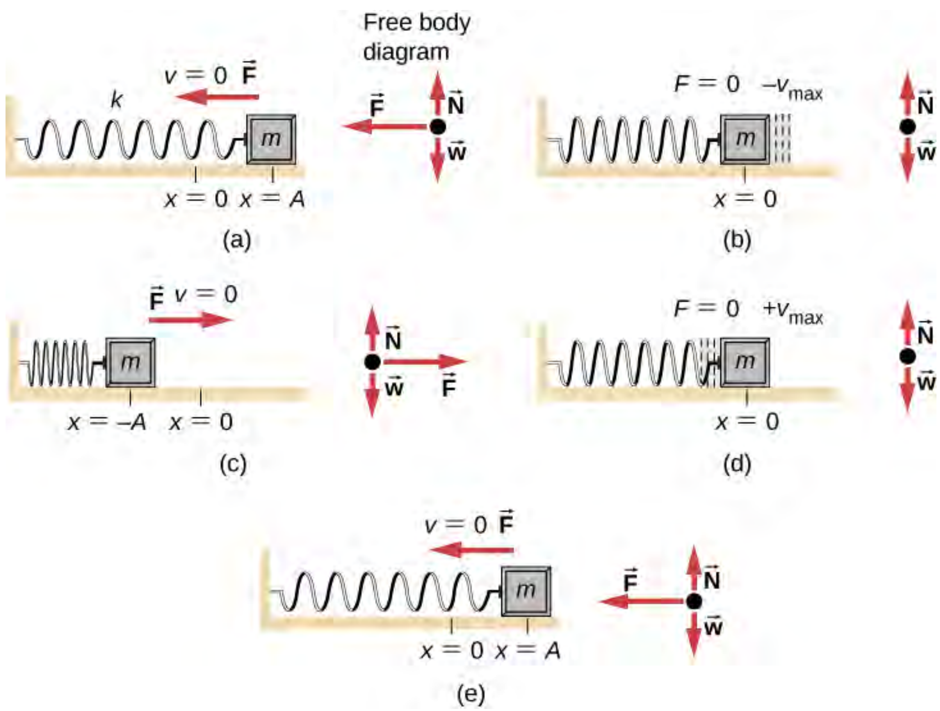


Figure 15.3 An object attached to a spring sliding on a frictionless surface is an uncomplicated simple harmonic oscillator. In the above set of figures, a mass is attached to a spring and placed on a frictionless table. The other end of the spring is attached to the wall. The position of the mass, when the spring is neither stretched nor compressed, is marked as $x = 0$ and is the equilibrium position. (a) The mass is displaced to a position $x = A$ and released from rest. (b) The mass accelerates as it moves in the negative *x*-direction, reaching a maximum negative velocity at $x = 0$. (c) The mass continues to move in the negative *x*-direction, slowing until it comes to a stop at $x = -A$. (d) The mass now begins to accelerate in the positive *x*-direction, reaching a positive maximum velocity at $x = 0$. (e) The mass then continues to move in the positive direction until it stops at $x = A$. The mass continues in SHM that has an amplitude *A* and a period *T*. The object's maximum speed occurs as it passes through equilibrium. The stiffer the spring is, the smaller the period *T*. The greater the mass of the object is, the greater the period *T*.

What is so significant about SHM? For one thing, the period *T* and frequency *f* of a simple harmonic oscillator are independent of amplitude. The string of a guitar, for example, oscillates with the same frequency whether plucked gently or hard.

Two important factors do affect the period of a simple harmonic oscillator. The period is related to how stiff the system is. A very stiff object has a large **force constant (*k*)**, which causes the system to have a smaller period. For example, you can adjust a diving board's stiffness—the stiffer it is, the faster it vibrates, and the shorter its period. Period also depends on the mass of the oscillating system. The more massive the system is, the longer the period. For example, a heavy person on a diving board bounces up and down more slowly than a light one. In fact, the mass *m* and the force constant *k* are the *only* factors that affect the period and frequency of SHM. To derive an equation for the period and the frequency, we must first define and analyze the equations of motion. Note that the force constant is sometimes referred to as the *spring constant*.

Equations of SHM

Consider a block attached to a spring on a frictionless table (Figure 15.4). The **equilibrium position** (the position where the spring is neither stretched nor compressed) is marked as $x = 0$. At the equilibrium position, the net force is zero.

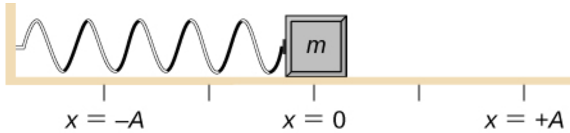


Figure 15.4 A block is attached to a spring and placed on a frictionless table. The equilibrium position, where the spring is neither extended nor compressed, is marked as $x = 0$.

Work is done on the block to pull it out to a position of $x = +A,$ and it is then released from rest. The maximum *x*-position (*A*) is called the amplitude of the motion. The block begins to oscillate in SHM between $x = +A$ and $x = -A,$ where *A* is the amplitude of the motion and *T* is the period of the oscillation. The period is the time for one oscillation. **Figure 15.5** shows the motion of the block as it completes one and a half oscillations after release. **Figure 15.6** shows a plot of the position of the block versus time. When the position is plotted versus time, it is clear that the data can be modeled by a cosine function with an amplitude *A* and a period *T*. The cosine function $\cos\theta$ repeats every multiple of $2\pi,$ whereas the motion of the block repeats every period *T*. However, the function $\cos\left(\frac{2\pi}{T}t\right)$ repeats every integer multiple of the period. The maximum of the cosine function is one, so it is necessary to multiply the cosine function by the amplitude *A*.

$$x(t) = A\cos\left(\frac{2\pi}{T}t\right) = A\cos(\omega t). \quad (15.2)$$

Recall from the chapter on rotation that the angular frequency equals $\omega = \frac{d\theta}{dt}$. In this case, the period is constant, so the angular frequency is defined as 2π divided by the period, $\omega = \frac{2\pi}{T}$.

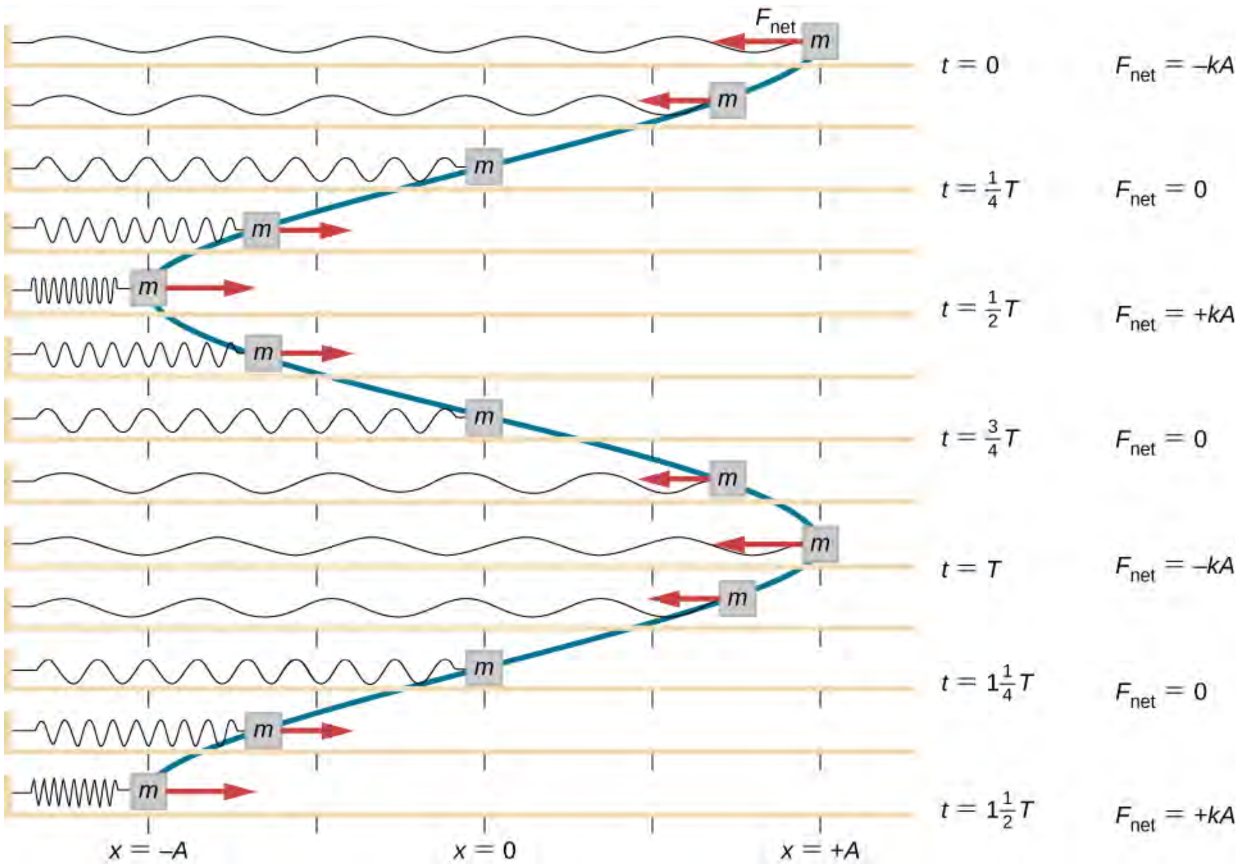


Figure 15.5 A block is attached to one end of a spring and placed on a frictionless table. The other end of the spring is anchored to the wall. The equilibrium position, where the net force equals zero, is marked as $x = 0\,\text{m}$. Work is done on the block, pulling it out to $x = +A$, and the block is released from rest. The block oscillates between $x = +A$ and $x = -A$. The force is also shown as a vector.

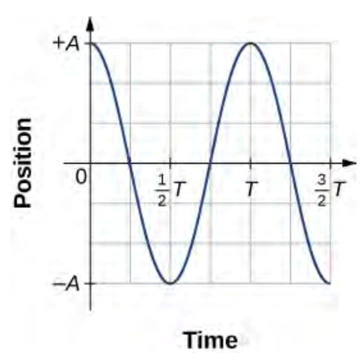


Figure 15.6 A graph of the position of the block shown in **Figure 15.5** as a function of time. The position can be modeled as a periodic function, such as a cosine or sine function.

The equation for the position as a function of time $x(t) = A\cos(\omega t)$ is good for modeling data, where the position of the block at the initial time $t = 0.00\text{ s}$ is at the amplitude A and the initial velocity is zero. Often when taking experimental data, the position of the mass at the initial time $t = 0.00\text{ s}$ is not equal to the amplitude and the initial velocity is not zero. Consider 10 seconds of data collected by a student in lab, shown in **Figure 15.7**.

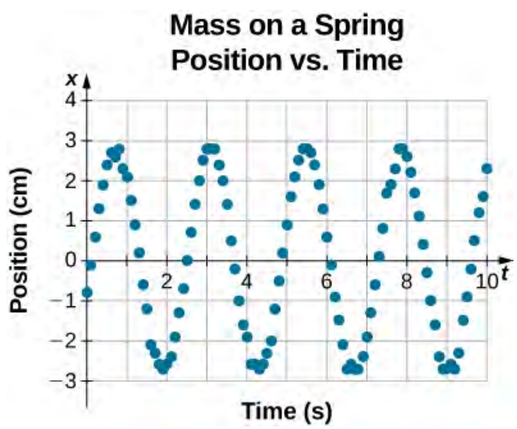


Figure 15.7 Data collected by a student in lab indicate the position of a block attached to a spring, measured with a sonic range finder. The data are collected starting at time $t = 0.00\text{s}$, but the initial position is near position $x \approx -0.80\,\text{cm} \neq 3.00\,\text{cm}$, so the initial position does not equal the amplitude $x_0 = +A$. The velocity is the time derivative of the position, which is the slope at a point on the graph of position versus time. The velocity is not $v = 0.00\,\text{m/s}$ at time $t = 0.00\,\text{s}$, as evident by the slope of the graph of position versus time, which is not zero at the initial time.

The data in **Figure 15.7** can still be modeled with a periodic function, like a cosine function, but the function is shifted to the right. This shift is known as a **phase shift** and is usually represented by the Greek letter phi (ϕ). The equation of the position as a function of time for a block on a spring becomes

$$x(t) = A\cos(\omega t + \phi).$$

This is the generalized equation for SHM where t is the time measured in seconds, ω is the angular frequency with units of inverse seconds, A is the amplitude measured in meters or centimeters, and ϕ is the phase shift measured in radians (**Figure 15.8**). It should be noted that because sine and cosine functions differ only by a phase shift, this motion could be modeled using either the cosine or sine function.

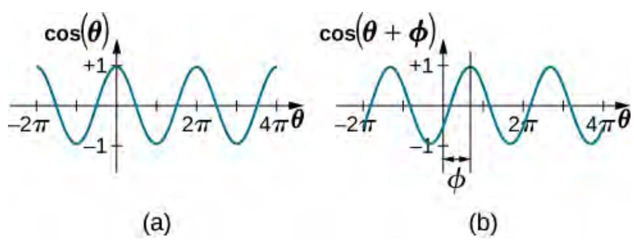


Figure 15.8 (a) A cosine function. (b) A cosine function shifted to the right by an angle ϕ. The angle ϕ is known as the phase shift of the function.

The velocity of the mass on a spring, oscillating in SHM, can be found by taking the derivative of the position equation:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(A\cos(\omega t + \phi)) = -A\omega\sin(\omega t + \varphi) = -v_{\max}\sin(\omega t + \phi).$$

Because the sine function oscillates between –1 and +1, the maximum velocity is the amplitude times the angular frequency, $v_{\max} = A\omega$. The maximum velocity occurs at the equilibrium position $(x = 0)$ when the mass is moving toward $x = +A$. The maximum velocity in the negative direction is attained at the equilibrium position $(x = 0)$ when the mass is moving toward $x = -A$ and is equal to $-v_{\max}$.

The acceleration of the mass on the spring can be found by taking the time derivative of the velocity:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(-A\omega \sin(\omega t + \phi)) = -A\omega^2 \cos(\omega t + \varphi) = -a_{max} \cos(\omega t + \phi).$$

The maximum acceleration is $a_{max} = A\omega^2$. The maximum acceleration occurs at the position $(x = -A)$, and the acceleration at the position $(x = -A)$ and is equal to $-a_{max}$.

Summary of Equations of Motion for SHM

In summary, the oscillatory motion of a block on a spring can be modeled with the following equations of motion:

$$x(t) = A\cos(\omega t + \phi) \quad (15.3)$$

$$v(t) = -v_{max} \sin(\omega t + \phi) \quad (15.4)$$

$$a(t) = -a_{max} \cos(\omega t + \phi) \quad (15.5)$$

$$x_{max} = A \quad (15.6)$$

$$v_{max} = A\omega \quad (15.7)$$

$$a_{max} = A\omega^2. \quad (15.8)$$

Here, A is the amplitude of the motion, T is the period, ϕ is the phase shift, and $\omega = \frac{2\pi}{T} = 2\pi f$ is the angular frequency of the motion of the block.

Example 15.2

Determining the Equations of Motion for a Block and a Spring

A 2.00-kg block is placed on a frictionless surface. A spring with a force constant of $k = 32.00\ \text{N/m}$ is attached to the block, and the opposite end of the spring is attached to the wall. The spring can be compressed or extended. The equilibrium position is marked as $x = 0.00\ \text{m}$.

Work is done on the block, pulling it out to $x = +0.02\ \text{m}$. The block is released from rest and oscillates between $x = +0.02\ \text{m}$ and $x = -0.02\ \text{m}$. The period of the motion is 1.57 s. Determine the equations of motion.

Strategy

We first find the angular frequency. The phase shift is zero, $\phi = 0.00\ \text{rad}$, because the block is released from rest at $x = A = +0.02\ \text{m}$. Once the angular frequency is found, we can determine the maximum velocity and maximum acceleration.

Solution

The angular frequency can be found and used to find the maximum velocity and maximum acceleration:

$$\begin{aligned} \omega &= \frac{2\pi}{1.57\ \text{s}} = 4.00\ \text{s}^{-1}; \\ v_{max} &= A\omega = 0.02\text{m}\left(4.00\ \text{s}^{-1}\right) = 0.08\ \text{m/s}; \\ a_{max} &= A\omega^2 = 0.02\ \text{m}\left(4.00\ \text{s}^{-1}\right)^2 = 0.32\ \text{m/s}^2. \end{aligned}$$

All that is left is to fill in the equations of motion:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) = (0.02\ \text{m})\cos\left(4.00\ \text{s}^{-1}\, t\right); \\ v(t) &= -v_{max} \sin(\omega t + \phi) = (-0.08\ \text{m/s})\sin\left(4.00\ \text{s}^{-1}\, t\right); \\ a(t) &= -a_{max} \cos(\omega t + \phi) = \left(-0.32\ \text{m/s}^2\right)\cos\left(4.00\ \text{s}^{-1}\, t\right). \end{aligned}$$

Significance

The position, velocity, and acceleration can be found for any time. It is important to remember that when using these equations, your calculator must be in radians mode.

The Period and Frequency of a Mass on a Spring

One interesting characteristic of the SHM of an object attached to a spring is that the angular frequency, and therefore the period and frequency of the motion, depend on only the mass and the force constant, and not on other factors such as the amplitude of the motion. We can use the equations of motion and Newton's second law $\left(\vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}}\right)$ to find equations for the angular frequency, frequency, and period.

Consider the block on a spring on a frictionless surface. There are three forces on the mass: the weight, the normal force, and the force due to the spring. The only two forces that act perpendicular to the surface are the weight and the normal force, which have equal magnitudes and opposite directions, and thus sum to zero. The only force that acts parallel to the surface is the force due to the spring, so the net force must be equal to the force of the spring:

$$\begin{aligned} F_x &= -kx; \\ ma &= -kx; \\ m\frac{d^2x}{dt^2} &= -kx; \\ \frac{d^2x}{dt^2} &= -\frac{k}{m}x. \end{aligned}$$

Substituting the equations of motion for x and a gives us

$$-A\omega^2\cos(\omega t + \phi) = -\frac{k}{m}A\cos(\omega t + \phi).$$

Cancelling out like terms and solving for the angular frequency yields

$$\omega = \sqrt{\frac{k}{m}}. \tag{15.9}$$

The angular frequency depends only on the force constant and the mass, and not the amplitude. The angular frequency is defined as $\omega = 2\pi/T$, which yields an equation for the period of the motion:

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{15.10}$$

The period also depends only on the mass and the force constant. The greater the mass, the longer the period. The stiffer the spring, the shorter the period. The frequency is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{15.11}$$

Vertical Motion and a Horizontal Spring

When a spring is hung vertically and a block is attached and set in motion, the block oscillates in SHM. In this case, there is no normal force, and the net effect of the force of gravity is to change the equilibrium position. Consider **Figure 15.9**. Two forces act on the block: the weight and the force of the spring. The weight is constant and the force of the spring changes as the length of the spring changes.

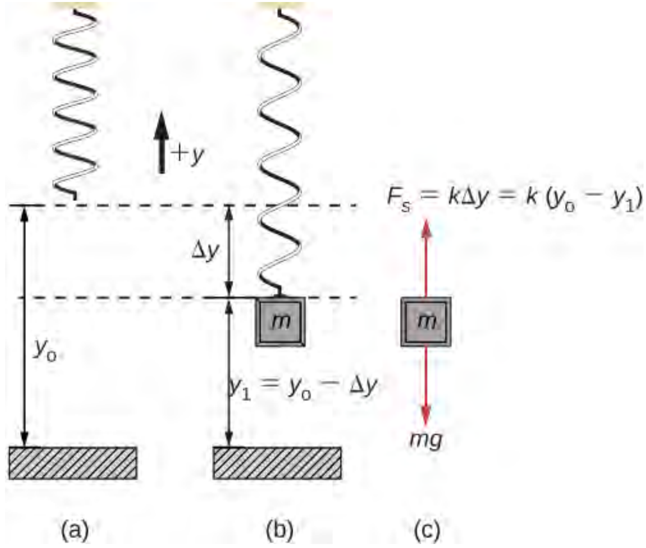


Figure 15.9 A spring is hung from the ceiling. When a block is attached, the block is at the equilibrium position where the weight of the block is equal to the force of the spring. (a) The spring is hung from the ceiling and the equilibrium position is marked as y_o. (b) A mass is attached to the spring and a new equilibrium position is reached ($y_1 = y_o - \Delta y$) when the force provided by the spring equals the weight of the mass. (c) The free-body diagram of the mass shows the two forces acting on the mass: the weight and the force of the spring.

When the block reaches the equilibrium position, as seen in **Figure 15.9**, the force of the spring equals the weight of the block, $F_{\text{net}} = F_s - mg = 0$, where

$$-k(-\Delta y) = mg.$$

From the figure, the change in the position is $\Delta y = y_0 - y_1$ and since $-k(-\Delta y) = mg$, we have

$$k(y_0 - y_1) - mg = 0.$$

If the block is displaced and released, it will oscillate around the new equilibrium position. As shown in **Figure 15.10**, if the position of the block is recorded as a function of time, the recording is a periodic function.

If the block is displaced to a position *y*, the net force becomes $F_{\text{net}} = k(y - y_0) - mg = 0$. But we found that at the equilibrium position, $mg = k\Delta y = ky_0 - ky_1$. Substituting for the weight in the equation yields

$$F_{\text{net}} = ky - ky_0 - (ky_0 - ky_1) = -k(y - y_1).$$

Recall that y_1 is just the equilibrium position and any position can be set to be the point $y = 0.00\text{m}$. So let's set y_1 to $y = 0.00\text{ m}$. The net force then becomes

$$F_{\text{net}} = -ky;$$
$$m\frac{d^2y}{dt^2} = -ky.$$

This is just what we found previously for a horizontally sliding mass on a spring. The constant force of gravity only served to shift the equilibrium location of the mass. Therefore, the solution should be the same form as for a block on a horizontal spring, $y(t) = A\cos(\omega t + \phi)$. The equations for the velocity and the acceleration also have the same form as for the horizontal case. Note that the inclusion of the phase shift means that the motion can actually be modeled using either a cosine or a sine function, since these two functions only differ by a phase shift.

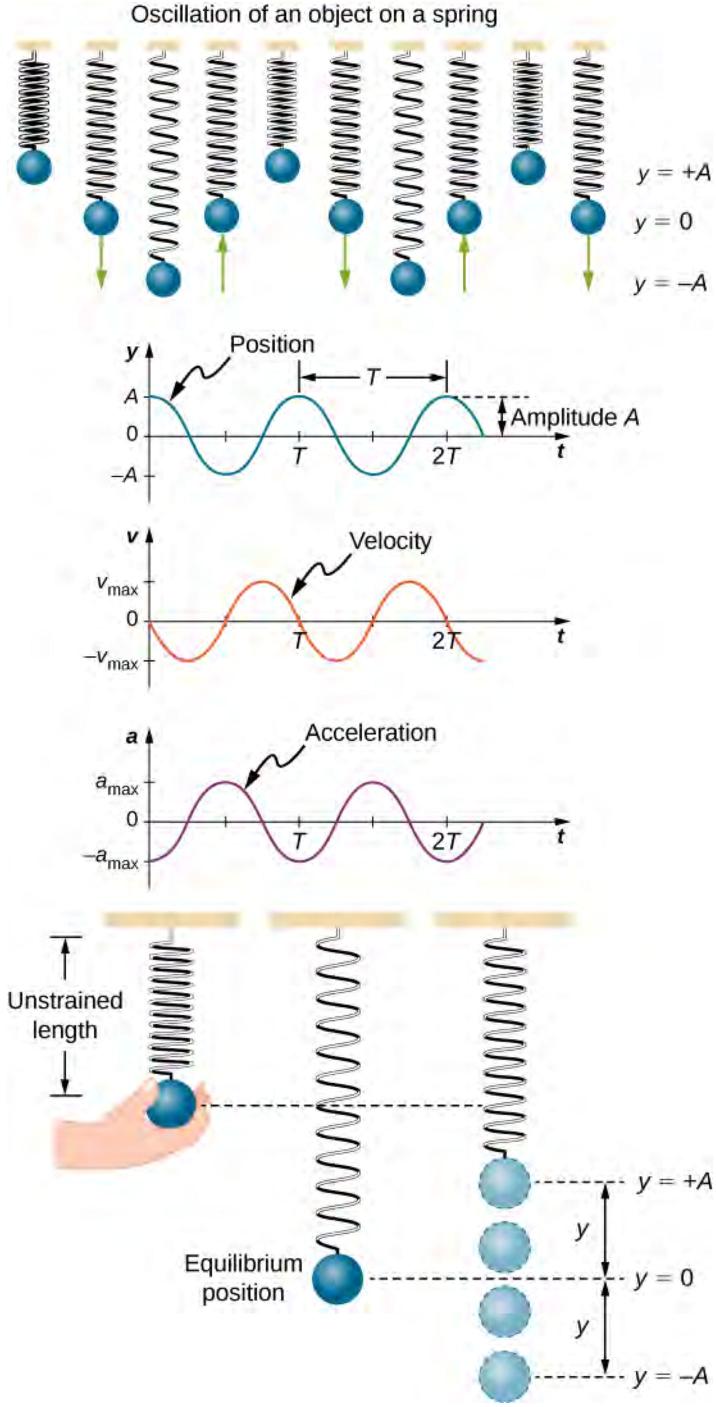


Figure 15.10 Graphs of $y(t)$, $v(t)$, and $a(t)$ versus t for the motion of an object on a vertical spring. The net force on the object can be described by Hooke's law, so the object undergoes SHM. Note that the initial position has the vertical displacement at its maximum value A; v is initially zero and then negative as the object moves down; the initial acceleration is negative, back toward the equilibrium position and becomes zero at that point.

15.2 | Energy in Simple Harmonic Motion

Learning Objectives

By the end of this section, you will be able to:

- Describe the energy conservation of the system of a mass and a spring
- Explain the concepts of stable and unstable equilibrium points

To produce a deformation in an object, we must do work. That is, whether you pluck a guitar string or compress a car's shock absorber, a force must be exerted through a distance. If the only result is deformation, and no work goes into thermal, sound, or kinetic energy, then all the work is initially stored in the deformed object as some form of potential energy.

Consider the example of a block attached to a spring on a frictionless table, oscillating in SHM. The force of the spring is a conservative force (which you studied in the chapter on potential energy and conservation of energy), and we can define a potential energy for it. This potential energy is the energy stored in the spring when the spring is extended or compressed. In this case, the block oscillates in one dimension with the force of the spring acting parallel to the motion:

$$W = \int_{x_i}^{x_f} F_x dx = \int_{x_i}^{x_f} -kx dx = \left[-\frac{1}{2}kx^2\right]_{x_i}^{x_f} = -\left[\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right] = -\left[U_f - U_i\right] = -\Delta U.$$

When considering the energy stored in a spring, the equilibrium position, marked as $x_i = 0.00\text{ m}$, is the position at which the energy stored in the spring is equal to zero. When the spring is stretched or compressed a distance *x*, the potential energy stored in the spring is

$$U = \frac{1}{2}kx^2.$$

Energy and the Simple Harmonic Oscillator

To study the energy of a simple harmonic oscillator, we need to consider all the forms of energy. Consider the example of a block attached to a spring, placed on a frictionless surface, oscillating in SHM. The potential energy stored in the deformation of the spring is

$$U = \frac{1}{2}kx^2.$$

In a simple harmonic oscillator, the energy oscillates between kinetic energy of the mass $K = \frac{1}{2}mv^2$ and potential energy $U = \frac{1}{2}kx^2$ stored in the spring. In the SHM of the mass and spring system, there are no dissipative forces, so the total energy is the sum of the potential energy and kinetic energy. In this section, we consider the conservation of energy of the system. The concepts examined are valid for all simple harmonic oscillators, including those where the gravitational force plays a role.

Consider **Figure 15.11**, which shows an oscillating block attached to a spring. In the case of undamped SHM, the energy oscillates back and forth between kinetic and potential, going completely from one form of energy to the other as the system oscillates. So for the simple example of an object on a frictionless surface attached to a spring, the motion starts with all of the energy stored in the spring as **elastic potential energy**. As the object starts to move, the elastic potential energy is converted into kinetic energy, becoming entirely kinetic energy at the equilibrium position. The energy is then converted back into elastic potential energy by the spring as it is stretched or compressed. The velocity becomes zero when the kinetic energy is completely converted, and this cycle then repeats. Understanding the conservation of energy in these cycles will provide extra insight here and in later applications of SHM, such as alternating circuits.

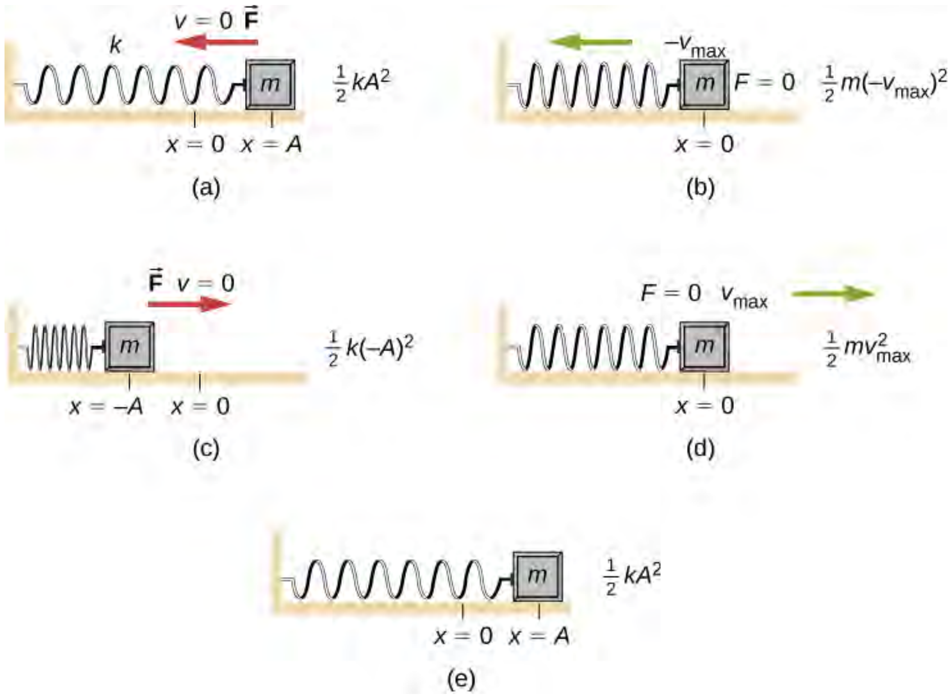


Figure 15.11 The transformation of energy in SHM for an object attached to a spring on a frictionless surface. (a) When the mass is at the position $x = +A$, all the energy is stored as potential energy in the spring $U = \frac{1}{2}kA^2$. The kinetic energy is equal to zero because the velocity of the mass is zero. (b) As the mass moves toward $x = -A$, the mass crosses the position $x = 0$. At this point, the spring is neither extended nor compressed, so the potential energy stored in the spring is zero. At $x = 0$, the total energy is all kinetic energy where $K = \frac{1}{2}m(-v_{\text{max}})^2$. (c) The mass continues to move until it reaches $x = -A$ where the mass stops and starts moving toward $x = +A$. At the position $x = -A$, the total energy is stored as potential energy in the compressed $U = \frac{1}{2}k(-A)^2$ and the kinetic energy is zero. (d) As the mass passes through the position $x = 0$, the kinetic energy is $K = \frac{1}{2}mv_{\text{max}}^2$ and the potential energy stored in the spring is zero. (e) The mass returns to the position $x = +A$, where $K = 0$ and $U = \frac{1}{2}kA^2$.

Consider **Figure 15.11**, which shows the energy at specific points on the periodic motion. While staying constant, the energy oscillates between the kinetic energy of the block and the potential energy stored in the spring:

$$E_{\text{Total}} = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2.$$

The motion of the block on a spring in SHM is defined by the position $x(t) = A\cos(\omega t + \phi)$ with a velocity of $v(t) = -A\omega\sin(\omega t + \phi)$. Using these equations, the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ and $\omega = \sqrt{\frac{k}{m}}$, we can find the total energy of the system:

$$\begin{aligned} E_{\text{Total}} &= \frac{1}{2}kA^2\cos^2(\omega t+\phi)+\frac{1}{2}mA^2\omega^2\sin^2(\omega t+\phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t+\phi)+\frac{1}{2}mA^2\left(\frac{k}{m}\right)\sin^2(\omega t+\phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t+\phi)+\frac{1}{2}kA^2\sin^2(\omega t+\phi) \\ &= \frac{1}{2}kA^2\left(\cos^2(\omega t+\phi)+\sin^2(\omega t+\phi)\right) \\ &= \frac{1}{2}kA^2. \end{aligned}$$

The total energy of the system of a block and a spring is equal to the sum of the potential energy stored in the spring plus the kinetic energy of the block and is proportional to the square of the amplitude $E_{\text{Total}} = (1/2)kA^2.$ The total energy of the system is constant.

A closer look at the energy of the system shows that the kinetic energy oscillates like a sine-squared function, while the potential energy oscillates like a cosine-squared function. However, the total energy for the system is constant and is proportional to the amplitude squared. **Figure 15.12** shows a plot of the potential, kinetic, and total energies of the block and spring system as a function of time. Also plotted are the position and velocity as a function of time. Before time $t = 0.0\text{ s},$ the block is attached to the spring and placed at the equilibrium position. Work is done on the block by applying an external force, pulling it out to a position of $x = +A$. The system now has potential energy stored in the spring. At time $t = 0.00\text{ s},$ the position of the block is equal to the amplitude, the potential energy stored in the spring is equal to $U = \frac{1}{2}kA^2$, and the force on the block is maximum and points in the negative *x*-direction $(F_S = -kA)$. The velocity and kinetic energy of the block are zero at time $t = 0.00\text{ s}.$ At time $t = 0.00\text{ s},$ the block is released from rest.

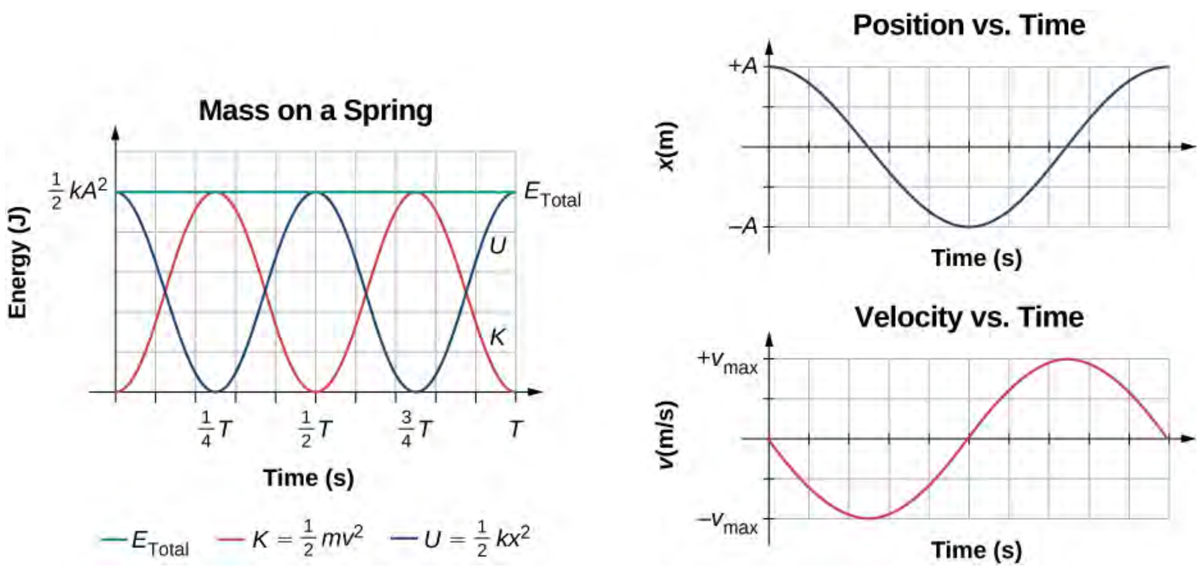


Figure 15.12 Graph of the kinetic energy, potential energy, and total energy of a block oscillating on a spring in SHM. Also shown are the graphs of position versus time and velocity versus time. The total energy remains constant, but the energy oscillates between kinetic energy and potential energy. When the kinetic energy is maximum, the potential energy is zero. This occurs when the velocity is maximum and the mass is at the equilibrium position. The potential energy is maximum when the speed is zero. The total energy is the sum of the kinetic energy plus the potential energy and it is constant.

Oscillations About an Equilibrium Position

We have just considered the energy of SHM as a function of time. Another interesting view of the simple harmonic oscillator is to consider the energy as a function of position. **Figure 15.13** shows a graph of the energy versus position of a system undergoing SHM.

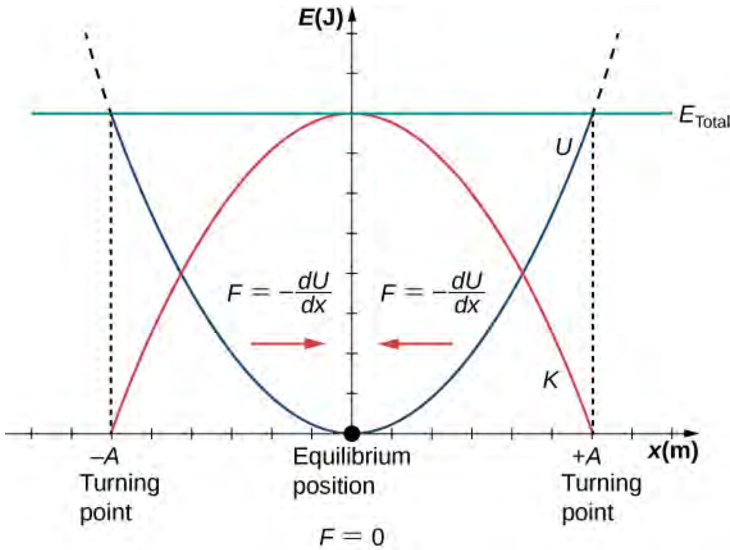


Figure 15.13 A graph of the kinetic energy (red), potential energy (blue), and total energy (green) of a simple harmonic oscillator. The force is equal to $F = -\frac{dU}{dx}$. The equilibrium position is shown as a black dot and is the point where the force is equal to zero. The force is positive when $x < 0$, negative when $x > 0$, and equal to zero when $x = 0$.

The potential energy curve in **Figure 15.13** resembles a bowl. When a marble is placed in a bowl, it settles to the equilibrium position at the lowest point of the bowl $(x = 0)$. This happens because a **restoring force** points toward the equilibrium point. This equilibrium point is sometimes referred to as a *fixed point*. When the marble is disturbed to a different position $(x = +A)$, the marble oscillates around the equilibrium position. Looking back at the graph of potential energy, the force can be found by looking at the slope of the potential energy graph $\left(F = -\frac{dU}{dx}\right)$. Since the force on either side of the fixed point points back toward the equilibrium point, the equilibrium point is called a **stable equilibrium point**. The points $x = A$ and $x = -A$ are called the turning points. (See **Potential Energy and Conservation of Energy**.)

Stability is an important concept. If an equilibrium point is stable, a slight disturbance of an object that is initially at the stable equilibrium point will cause the object to oscillate around that point. The stable equilibrium point occurs because the force on either side is directed toward it. For an unstable equilibrium point, if the object is disturbed slightly, it does not return to the equilibrium point.

Consider the marble in the bowl example. If the bowl is right-side up, the marble, if disturbed slightly, will oscillate around the stable equilibrium point. If the bowl is turned upside down, the marble can be balanced on the top, at the equilibrium point where the net force is zero. However, if the marble is disturbed slightly, it will not return to the equilibrium point, but will instead roll off the bowl. The reason is that the force on either side of the equilibrium point is directed away from that point. This point is an unstable equilibrium point.

Figure 15.14 shows three conditions. The first is a stable equilibrium point (a), the second is an unstable equilibrium point (b), and the last is also an unstable equilibrium point (c), because the force on only one side points toward the equilibrium point.

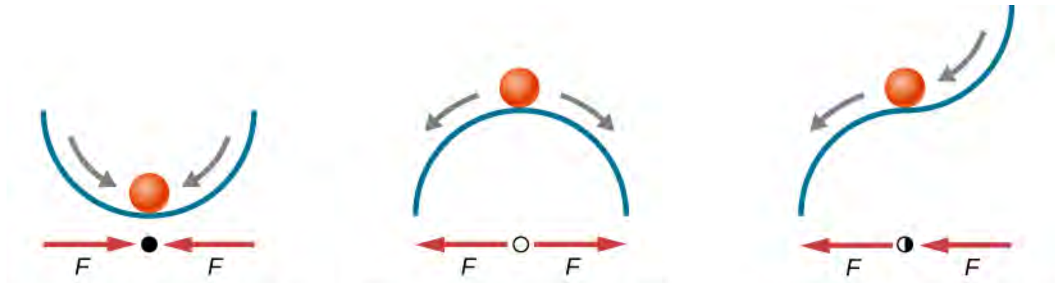


Figure 15.14 Examples of equilibrium points. (a) Stable equilibrium point; (b) unstable equilibrium point; (c) unstable equilibrium point (sometimes referred to as a half-stable equilibrium point).

The process of determining whether an equilibrium point is stable or unstable can be formalized. Consider the potential energy curves shown in **Figure 15.15**. The force can be found by analyzing the slope of the graph. The force is $F = -\frac{dU}{dx}$. In (a), the fixed point is at $x = 0.00\,\text{m}$. When $x < 0.00\,\text{m}$, the force is positive. When $x > 0.00\,\text{m}$, the force is negative. This is a stable point. In (b), the fixed point is at $x = 0.00\,\text{m}$. When $x < 0.00\,\text{m}$, the force is negative. When $x > 0.00\,\text{m}$, the force is also negative. This is an unstable point.

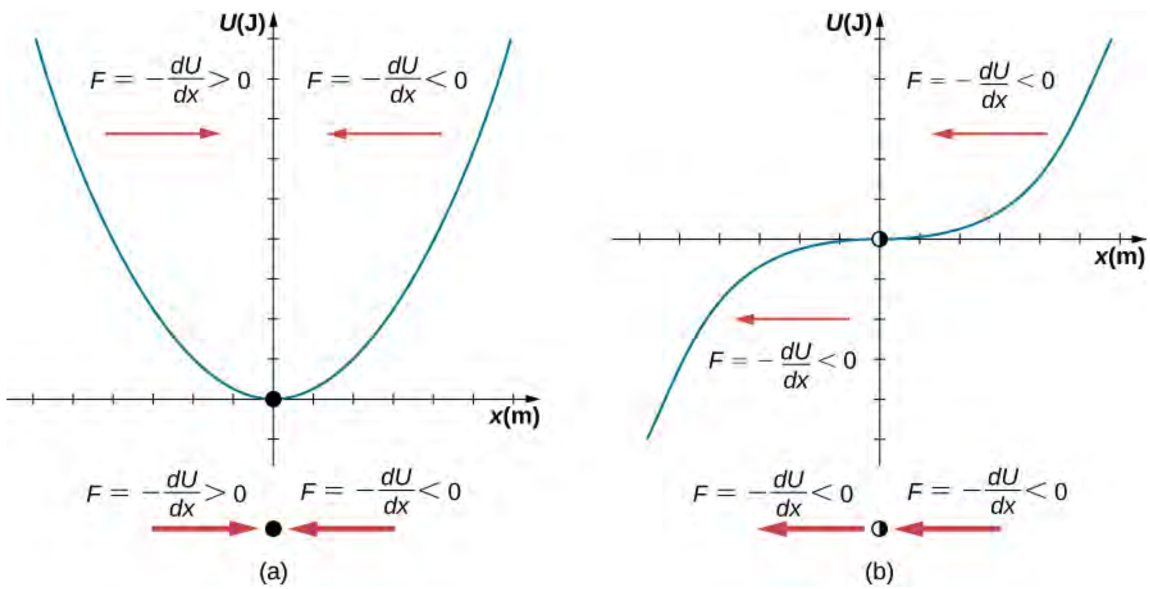


Figure 15.15 Two examples of a potential energy function. The force at a position is equal to the negative of the slope of the graph at that position. (a) A potential energy function with a stable equilibrium point. (b) A potential energy function with an unstable equilibrium point. This point is sometimes called half-stable because the force on one side points toward the fixed point.

A practical application of the concept of stable equilibrium points is the force between two neutral atoms in a molecule. If two molecules are in close proximity, separated by a few atomic diameters, they can experience an attractive force. If the molecules move close enough so that the electron shells of the other electrons overlap, the force between the molecules becomes repulsive. The attractive force between the two atoms may cause the atoms to form a molecule. The force between the two molecules is not a linear force and cannot be modeled simply as two masses separated by a spring, but the atoms of the molecule can oscillate around an equilibrium point when displaced a small amount from the equilibrium position. The atoms oscillate due the attractive force and repulsive force between the two atoms.

Consider one example of the interaction between two atoms known as the van Der Waals interaction. It is beyond the scope of this chapter to discuss in depth the interactions of the two atoms, but the oscillations of the atoms can be examined by considering one example of a model of the potential energy of the system. One suggestion to model the potential energy of this molecule is with the Lennard-Jones 6-12 potential:

$$U(x) = 4\varepsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right].$$

A graph of this function is shown in **Figure 15.16**. The two parameters ε and σ are found experimentally.

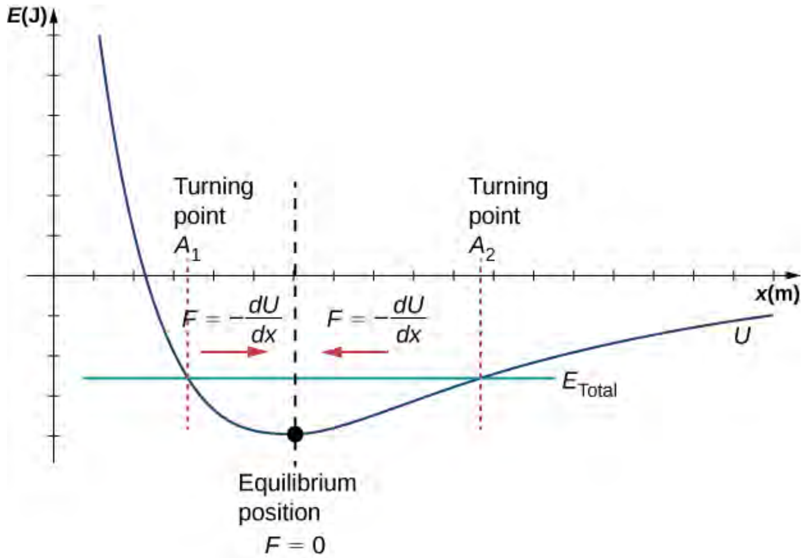


Figure 15.16 The Lennard-Jones potential energy function for a system of two neutral atoms. If the energy is below some maximum energy, the system oscillates near the equilibrium position between the two turning points.

From the graph, you can see that there is a potential energy well, which has some similarities to the potential energy well of the potential energy function of the simple harmonic oscillator discussed in **Figure 15.13**. The Lennard-Jones potential has a stable equilibrium point where the potential energy is minimum and the force on either side of the equilibrium point points toward equilibrium point. Note that unlike the simple harmonic oscillator, the potential well of the Lennard-Jones potential is not symmetric. This is due to the fact that the force between the atoms is not a Hooke's law force and is not linear. The atoms can still oscillate around the equilibrium position x_{min} because when $x < x_{\text{min}}$, the force is positive; when $x > x_{\text{min}}$, the force is negative. Notice that as *x* approaches zero, the slope is quite steep and negative, which means that the force is large and positive. This suggests that it takes a large force to try to push the atoms close together. As *x* becomes increasingly large, the slope becomes less steep and the force is smaller and negative. This suggests that if given a large enough energy, the atoms can be separated.

If you are interested in this interaction, find the force between the molecules by taking the derivative of the potential energy function. You will see immediately that the force does not resemble a Hooke's law force $(F = -kx)$, but if you are familiar with the binomial theorem:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots,$$

the force can be approximated by a Hooke's law force.

Velocity and Energy Conservation

Getting back to the system of a block and a spring in **Figure 15.11**, once the block is released from rest, it begins to move in the negative direction toward the equilibrium position. The potential energy decreases and the magnitude of the velocity and the kinetic energy increase. At time $t = T/4$, the block reaches the equilibrium position $x = 0.00\,\text{m}$, where the force on the block and the potential energy are zero. At the equilibrium position, the block reaches a negative velocity with a magnitude equal to the maximum velocity $v = -A\omega$. The kinetic energy is maximum and equal to $K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2 = \frac{1}{2}kA^2.$ At this point, the force on the block is zero, but momentum carries the block, and it continues in the negative direction toward $x = -A$. As the block continues to move, the force on it acts in the positive direction and the magnitude of the velocity and kinetic energy decrease. The potential energy increases as the spring compresses. At time $t = T/2$, the block reaches $x = -A$. Here the velocity and kinetic energy are equal to zero. The force on the block is $F = +kA$ and the potential energy stored in the spring is $U = \frac{1}{2}kA^2$. During the oscillations, the total energy is constant and equal to the sum of the potential energy and the kinetic energy of the system,

$$E_{\text{Total}} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2. \tag{15.12}$$

The equation for the energy associated with SHM can be solved to find the magnitude of the velocity at any position:

$$|v| = \sqrt{\frac{k}{m}(A^2 - x^2)}. \tag{15.13}$$

The energy in a simple harmonic oscillator is proportional to the square of the amplitude. When considering many forms of oscillations, you will find the energy proportional to the amplitude squared.

15.1 Check Your Understanding Why would it hurt more if you snapped your hand with a ruler than with a loose spring, even if the displacement of each system is equal?

15.2 Check Your Understanding Identify one way you could decrease the maximum velocity of a simple harmonic oscillator.

15.3 | Comparing Simple Harmonic Motion and Circular Motion

Learning Objectives

By the end of this section, you will be able to:

- Describe how the sine and cosine functions relate to the concepts of circular motion
- Describe the connection between simple harmonic motion and circular motion

An easy way to model SHM is by considering uniform circular motion. **Figure 15.17** shows one way of using this method. A peg (a cylinder of wood) is attached to a vertical disk, rotating with a constant angular frequency. **Figure 15.18** shows a side view of the disk and peg. If a lamp is placed above the disk and peg, the peg produces a shadow. Let the disk have a radius of $r = A$ and define the position of the shadow that coincides with the center line of the disk to be $x = 0.00\,m$. As the disk rotates at a constant rate, the shadow oscillates between $x = +A$ and $x = -A$. Now imagine a block on a spring beneath the floor as shown in **Figure 15.18**.

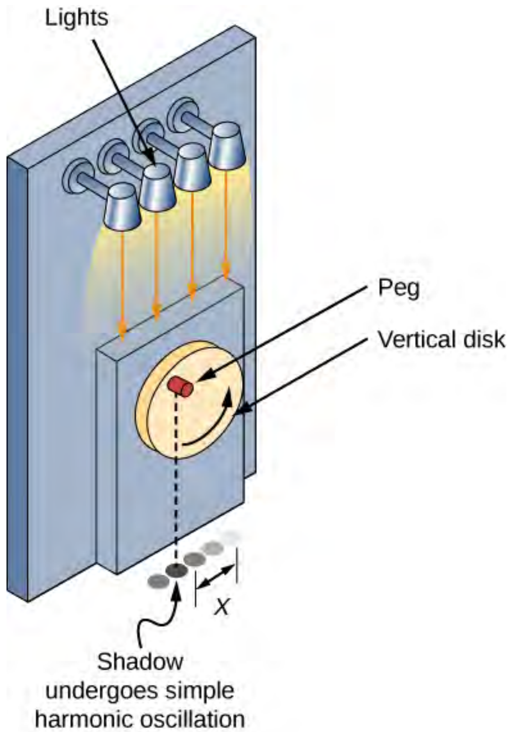


Figure 15.17 SHM can be modeled as rotational motion by looking at the shadow of a peg on a wheel rotating at a constant angular frequency.

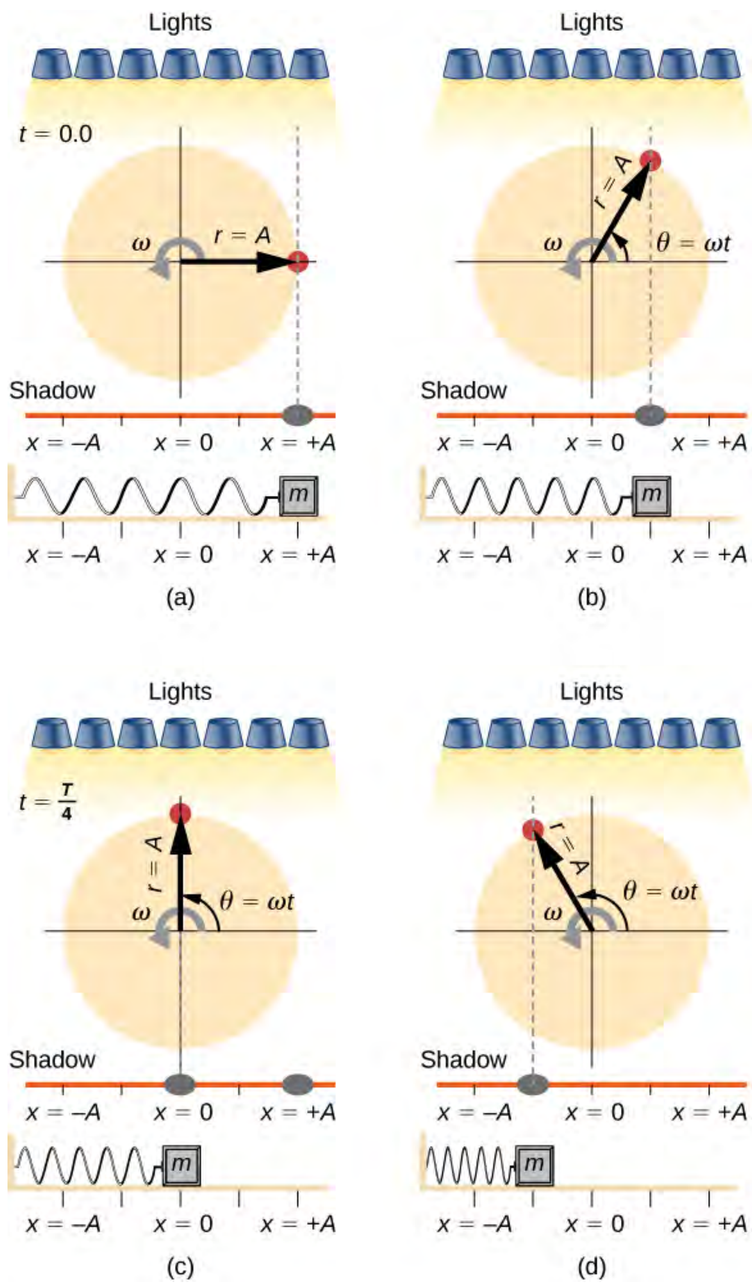


Figure 15.18 Light shines down on the disk so that the peg makes a shadow. If the disk rotates at just the right angular frequency, the shadow follows the motion of the block on a spring. If there is no energy dissipated due to nonconservative forces, the block and the shadow will oscillate back and forth in unison. In this figure, four snapshots are taken at four different times. (a) The wheel starts at $\theta = 0^o$ and the shadow of the peg is at $x = +A$, representing the mass at position $x = +A$. (b) As the disk rotates through an angle $\theta = \omega t$, the shadow of the peg is between $x = +A$ and $x = 0$. (c) The disk continues to rotate until $\theta = 90^0$, at which the shadow follows the mass to $x = 0$. (d) The disk continues to rotate, the shadow follows the position of the mass.

If the disk turns at the proper angular frequency, the shadow follows along with the block. The position of the shadow can be modeled with the equation

$$x(t) = A\cos(\omega t). \tag{15.14}$$

Recall that the block attached to the spring does not move at a constant velocity. How often does the wheel have to turn to have the peg's shadow always on the block? The disk must turn at a constant angular frequency equal to 2π times the frequency of oscillation $(\omega = 2\pi f)$.

Figure 15.19 shows the basic relationship between uniform circular motion and SHM. The peg lies at the tip of the radius, a distance A from the center of the disk. The x-axis is defined by a line drawn parallel to the ground, cutting the disk in half. The y-axis (not shown) is defined by a line perpendicular to the ground, cutting the disk into a left half and a right half. The center of the disk is the point $(x = 0,\ y = 0)$. The projection of the position of the peg onto the fixed x-axis gives the position of the shadow, which undergoes SHM analogous to the system of the block and spring. At the time shown in the figure, the projection has position x and moves to the left with velocity v. The tangential velocity of the peg around the circle equals $\bar{v}_{max}$ of the block on the spring. The x-component of the velocity is equal to the velocity of the block on the spring.

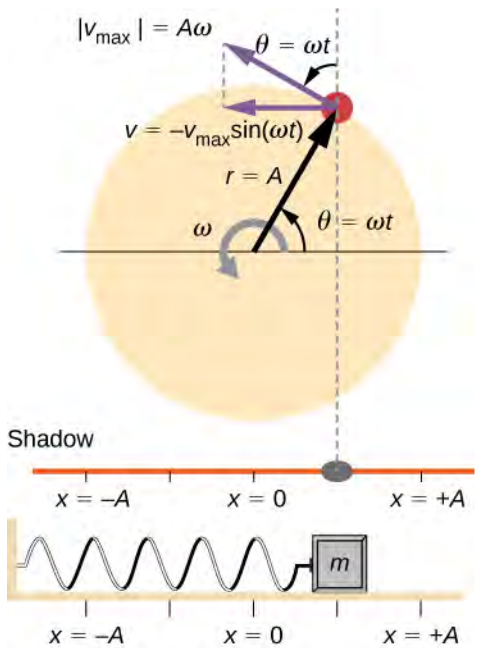


Figure 15.19 A peg moving on a circular path with a constant angular velocity ω is undergoing uniform circular motion. Its projection on the x-axis undergoes SHM. Also shown is the velocity of the peg around the circle, v_{max}, and its projection, which is v. Note that these velocities form a similar triangle to the displacement triangle.

We can use **Figure 15.19** to analyze the velocity of the shadow as the disk rotates. The peg moves in a circle with a speed of $v_{max} = A\omega$. The shadow moves with a velocity equal to the component of the peg's velocity that is parallel to the surface where the shadow is being produced:

$$v = -v_{max}\sin(\omega t). \tag{15.15}$$

It follows that the acceleration is

$$a = -a_{max} \cos(\omega t). \quad (15.16)$$

15.3 Check Your Understanding Identify an object that undergoes uniform circular motion. Describe how you could trace the SHM of this object.

15.4 | Pendulums

Learning Objectives

By the end of this section, you will be able to:

- State the forces that act on a simple pendulum
- Determine the angular frequency, frequency, and period of a simple pendulum in terms of the length of the pendulum and the acceleration due to gravity
- Define the period for a physical pendulum
- Define the period for a torsional pendulum

Pendulums are in common usage. Grandfather clocks use a pendulum to keep time and a pendulum can be used to measure the acceleration due to gravity. For small displacements, a pendulum is a simple harmonic oscillator.

The Simple Pendulum

A **simple pendulum** is defined to have a point mass, also known as the pendulum bob, which is suspended from a string of length L with negligible mass (Figure 15.20). Here, the only forces acting on the bob are the force of gravity (i.e., the weight of the bob) and tension from the string. The mass of the string is assumed to be negligible as compared to the mass of the bob.

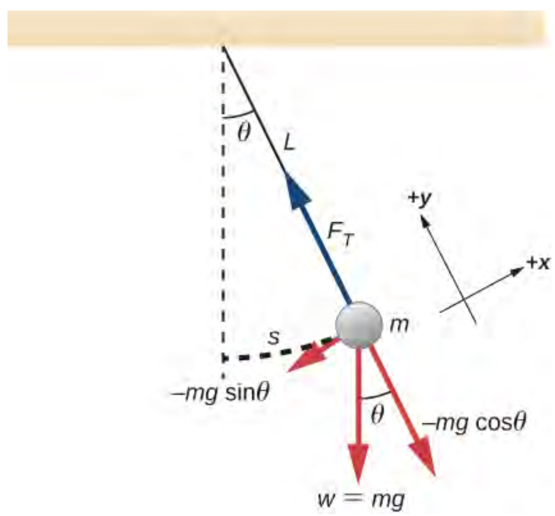


Figure 15.20 A simple pendulum has a small-diameter bob and a string that has a very small mass but is strong enough not to stretch appreciably. The linear displacement from equilibrium is s, the length of the arc. Also shown are the forces on the bob, which result in a net force of $-mg\sin\theta$ toward the equilibrium position—that is, a restoring force.

Consider the torque on the pendulum. The force providing the restoring torque is the component of the weight of the pendulum bob that acts along the arc length. The torque is the length of the string L times the component of the net force that is perpendicular to the radius of the arc. The minus sign indicates the torque acts in the opposite direction of the angular displacement:

$$\begin{aligned} \tau &= -L(mg\sin\theta); \\ I\alpha &= -L(mg\sin\theta); \\ I\frac{d^2\theta}{dt^2} &= -L(mg\sin\theta); \\ mL^2\frac{d^2\theta}{dt^2} &= -L(mg\sin\theta); \\ \frac{d^2\theta}{dt^2} &= -\frac{g}{L}\sin\theta. \end{aligned}$$

The solution to this differential equation involves advanced calculus, and is beyond the scope of this text. But note that for small angles (less than 15 degrees), $\sin\theta$ and θ differ by less than 1%, so we can use the small angle approximation $\sin\theta \approx \theta$. The angle θ describes the position of the pendulum. Using the small angle approximation gives an approximate solution for small angles,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta. \quad (15.17)$$

Because this equation has the same form as the equation for SHM, the solution is easy to find. The angular frequency is

$$\omega = \sqrt{\frac{g}{L}} \quad (15.18)$$

and the period is

$$T = 2\pi\sqrt{\frac{L}{g}}. \quad (15.19)$$

The period of a simple pendulum depends on its length and the acceleration due to gravity. The period is completely independent of other factors, such as mass and the maximum displacement. As with simple harmonic oscillators, the period T for a pendulum is nearly independent of amplitude, especially if θ is less than about $15°$. Even simple pendulum clocks can be finely adjusted and remain accurate.

Note the dependence of T on g. If the length of a pendulum is precisely known, it can actually be used to measure the acceleration due to gravity, as in the following example.

Example 15.3

Measuring Acceleration due to Gravity by the Period of a Pendulum

What is the acceleration due to gravity in a region where a simple pendulum having a length 75.000 cm has a period of 1.7357 s?

Strategy

We are asked to find g given the period T and the length L of a pendulum. We can solve $T = 2\pi\sqrt{\frac{L}{g}}$ for g, assuming only that the angle of deflection is less than $15°$.

Solution

1. Square $T = 2\pi\sqrt{\frac{L}{g}}$ and solve for g:

$$g = 4\pi^2 \frac{L}{T^2}.$$

2. Substitute known values into the new equation:

$$g = 4\pi^2 \frac{0.75000 \text{ m}}{(1.7357 \text{ s})^2}.$$

3. Calculate to find g:

$$g = 9.8281 \text{ m/s}^2.$$

Significance

This method for determining g can be very accurate, which is why length and period are given to five digits in this example. For the precision of the approximation $\sin\theta \approx \theta$ to be better than the precision of the pendulum length and period, the maximum displacement angle should be kept below about $0.5°$.

15.4 Check Your Understanding An engineer builds two simple pendulums. Both are suspended from small wires secured to the ceiling of a room. Each pendulum hovers 2 cm above the floor. Pendulum 1 has a bob with a mass of 10 kg. Pendulum 2 has a bob with a mass of 100 kg. Describe how the motion of the pendulums will differ if the bobs are both displaced by $12°$.

Physical Pendulum

Any object can oscillate like a pendulum. Consider a coffee mug hanging on a hook in the pantry. If the mug gets knocked, it oscillates back and forth like a pendulum until the oscillations die out. We have described a simple pendulum as a point mass and a string. A **physical pendulum** is any object whose oscillations are similar to those of the simple pendulum, but cannot be modeled as a point mass on a string, and the mass distribution must be included into the equation of motion.

As for the simple pendulum, the restoring force of the physical pendulum is the force of gravity. With the simple pendulum, the force of gravity acts on the center of the pendulum bob. In the case of the physical pendulum, the force of gravity acts on the center of mass (CM) of an object. The object oscillates about a point *O*. Consider an object of a generic shape as shown in **Figure 15.21**.

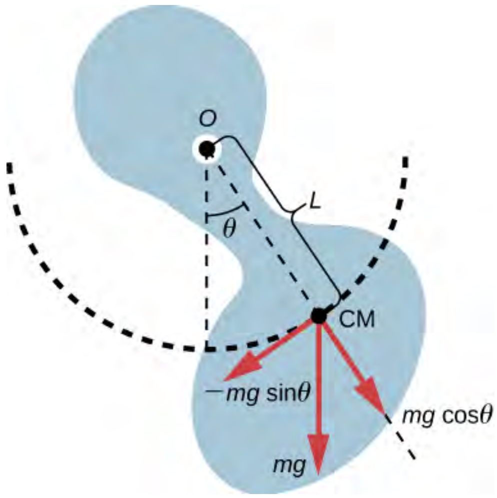


Figure 15.21 A physical pendulum is any object that oscillates as a pendulum, but cannot be modeled as a point mass on a string. The force of gravity acts on the center of mass (CM) and provides the restoring force that causes the object to oscillate. The minus sign on the component of the weight that provides the restoring force is present because the force acts in the opposite direction of the increasing angle θ.

When a physical pendulum is hanging from a point but is free to rotate, it rotates because of the torque applied at the CM, produced by the component of the object's weight that acts tangent to the motion of the CM. Taking the counterclockwise direction to be positive, the component of the gravitational force that acts tangent to the motion is $-mg\sin\theta$. The minus sign is the result of the restoring force acting in the opposite direction of the increasing angle. Recall that the torque is equal to $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$. The magnitude of the torque is equal to the length of the radius arm times the tangential component of the force applied, $|\tau| = rF\sin\theta$. Here, the length L of the radius arm is the distance between the point of rotation and the CM. To analyze the motion, start with the net torque. Like the simple pendulum, consider only small angles so that $\sin\theta \approx \theta$. Recall from **Fixed-Axis Rotation** on rotation that the net torque is equal to the moment of inertia $I = \int r^2\,dm$ times the angular acceleration α, where $\alpha = \frac{d^2\theta}{dt^2}$:

$$I\alpha = \tau_{\text{net}} = L(-mg)\sin\theta.$$

Using the small angle approximation and rearranging:

$$\begin{aligned} I\alpha &= -L(mg)\theta; \\ I\frac{d^2\theta}{dt^2} &= -L(mg)\theta; \\ \frac{d^2\theta}{dt^2} &= -\left(\frac{mgL}{I}\right)\theta. \end{aligned}$$

Once again, the equation says that the second time derivative of the position (in this case, the angle) equals minus a constant $\left(-\frac{mgL}{I}\right)$ times the position. The solution is

$$\theta(t) = \Theta\cos(\omega t + \phi),$$

where Θ is the maximum angular displacement. The angular frequency is

$$\omega = \sqrt{\frac{mgL}{I}}. \tag{15.20}$$

The period is therefore

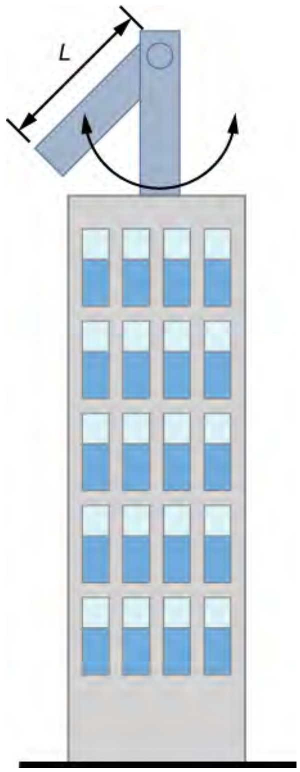
$$T = 2\pi\sqrt{\frac{I}{mgL}}. \tag{15.21}$$

Note that for a simple pendulum, the moment of inertia is $I = \int r^2\, dm = mL^2$ and the period reduces to $T = 2\pi\sqrt{\frac{L}{g}}$.

Example 15.4

Reducing the Swaying of a Skyscraper

In extreme conditions, skyscrapers can sway up to two meters with a frequency of up to 20.00 Hz due to high winds or seismic activity. Several companies have developed physical pendulums that are placed on the top of the skyscrapers. As the skyscraper sways to the right, the pendulum swings to the left, reducing the sway. Assuming the oscillations have a frequency of 0.50 Hz, design a pendulum that consists of a long beam, of constant density, with a mass of 100 metric tons and a pivot point at one end of the beam. What should be the length of the beam?



Strategy

We are asked to find the length of the physical pendulum with a known mass. We first need to find the moment of inertia of the beam. We can then use the equation for the period of a physical pendulum to find the length.

Solution

1. Find the moment of inertia for the CM:
2. Use the parallel axis theorem to find the moment of inertia about the point of rotation:

$$I = I_{\text{CM}} + \frac{L^2}{4}M = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2.$$

3. The period of a physical pendulum has a period of $T = 2\pi\sqrt{\frac{I}{mgL}}$. Use the moment of inertia to solve for the length L:

$$T = 2\pi\sqrt{\frac{I}{MgL}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL}} = 2\pi\sqrt{\frac{L}{3g}};$$

$$L = 3g\left(\frac{T}{2\pi}\right)^2 = 3\left(9.8\frac{\text{m}}{\text{s}^2}\right)\left(\frac{2\text{ s}}{2\pi}\right)^2 = 2.98\text{ m}.$$

Significance

There are many ways to reduce the oscillations, including modifying the shape of the skyscrapers, using multiple physical pendulums, and using tuned-mass dampers.

Torsional Pendulum

A **torsional pendulum** consists of a rigid body suspended by a light wire or spring (Figure 15.22). When the body is twisted some small maximum angle (Θ) and released from rest, the body oscillates between $(\theta = +\Theta)$ and $(\theta = -\Theta)$. The restoring torque is supplied by the shearing of the string or wire.

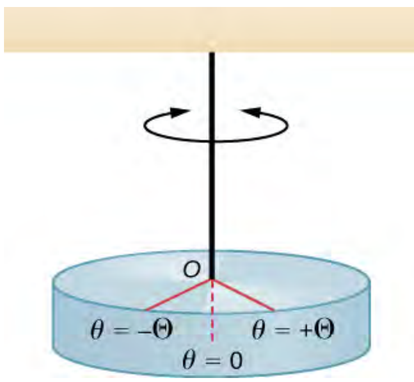


Figure 15.22 A torsional pendulum consists of a rigid body suspended by a string or wire. The rigid body oscillates between $\theta = +\Theta$ and $\theta = -\Theta$.

The restoring torque can be modeled as being proportional to the angle:

$$\tau = -\kappa\theta.$$

The variable kappa (κ) is known as the torsion constant of the wire or string. The minus sign shows that the restoring torque acts in the opposite direction to increasing angular displacement. The net torque is equal to the moment of inertia times the angular acceleration:

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta;$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta.$$

This equation says that the second time derivative of the position (in this case, the angle) equals a negative constant times the position. This looks very similar to the equation of motion for the SHM $\frac{d^2x}{dt^2} = -\frac{k}{m}x$, where the period was found to be $T = 2\pi\sqrt{\frac{m}{k}}$. Therefore, the period of the torsional pendulum can be found using

$$T = 2\pi\sqrt{\frac{I}{\kappa}}. \tag{15.22}$$

The units for the torsion constant are $[\kappa] = \text{N-m} = \left(\text{kg}\frac{\text{m}}{\text{s}^2}\right)\text{m} = \text{kg}\frac{\text{m}^2}{\text{s}^2}$ and the units for the moment of inertial are $[I] = \text{kg-m}^2,$ which show that the unit for the period is the second.

Example 15.5

Measuring the Torsion Constant of a String

A rod has a length of $l = 0.30\text{ m}$ and a mass of 4.00 kg. A string is attached to the CM of the rod and the system is hung from the ceiling (**Figure 15.23**). The rod is displaced 10 degrees from the equilibrium position and released from rest. The rod oscillates with a period of 0.5 s. What is the torsion constant κ?

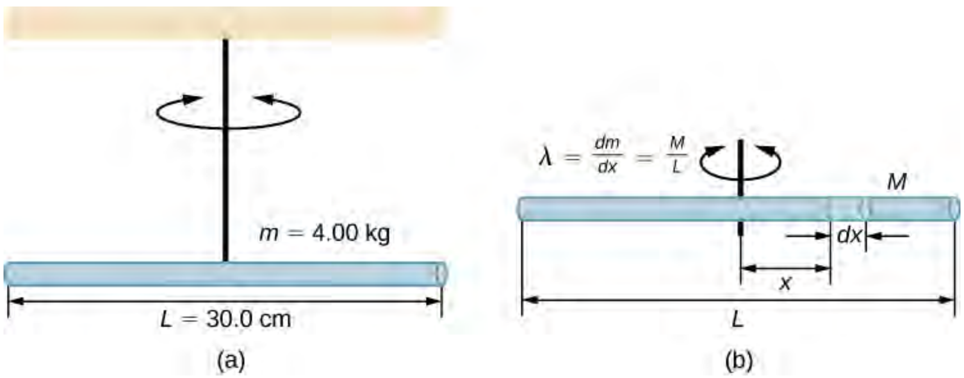


Figure 15.23 (a) A rod suspended by a string from the ceiling. (b) Finding the rod's moment of inertia.

Strategy

We are asked to find the torsion constant of the string. We first need to find the moment of inertia.

Solution

1. Find the moment of inertia for the CM:

$$I_{\text{CM}} = \int x^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda dx = \lambda\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2} = \lambda\frac{2L^3}{24} = \left(\frac{M}{L}\right)\frac{2L^3}{24} = \frac{1}{12}ML^2.$$

2. Calculate the torsion constant using the equation for the period:

$$\begin{aligned} T &= 2\pi\sqrt{\frac{I}{\kappa}}; \\ \kappa &= I\left(\frac{2\pi}{T}\right)^2 = \left(\frac{1}{12}ML^2\right)\left(\frac{2\pi}{T}\right)^2; \\ &= \left(\frac{1}{12}(4.00\text{ kg})(0.30\text{ m})^2\right)\left(\frac{2\pi}{0.50\text{ s}}\right)^2 = 4.73\text{ N}\cdot\text{m}. \end{aligned}$$

Significance

Like the force constant of the system of a block and a spring, the larger the torsion constant, the shorter the period.

15.5 | Damped Oscillations

Learning Objectives

By the end of this section, you will be able to:

- Describe the motion of damped harmonic motion
- Write the equations of motion for damped harmonic oscillations
- Describe the motion of driven, or forced, damped harmonic motion
- Write the equations of motion for forced, damped harmonic motion

In the real world, oscillations seldom follow true SHM. Friction of some sort usually acts to dampen the motion so it dies away, or needs more force to continue. In this section, we examine some examples of damped harmonic motion and see how to modify the equations of motion to describe this more general case.

A guitar string stops oscillating a few seconds after being plucked. To keep swinging on a playground swing, you must keep pushing (**Figure 15.24**). Although we can often make friction and other nonconservative forces small or negligible, completely undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.

Figure 15.24 To counteract dampening forces, you need to keep pumping a swing. (credit: Bob Mical)

Figure 15.25 shows a mass *m* attached to a spring with a force constant k. The mass is raised to a position A_0, the initial amplitude, and then released. The mass oscillates around the equilibrium position in a fluid with viscosity but the amplitude decreases for each oscillation. For a system that has a small amount of damping, the period and frequency are constant and are nearly the same as for SHM, but the amplitude gradually decreases as shown. This occurs because the non-conservative damping force removes energy from the system, usually in the form of thermal energy.

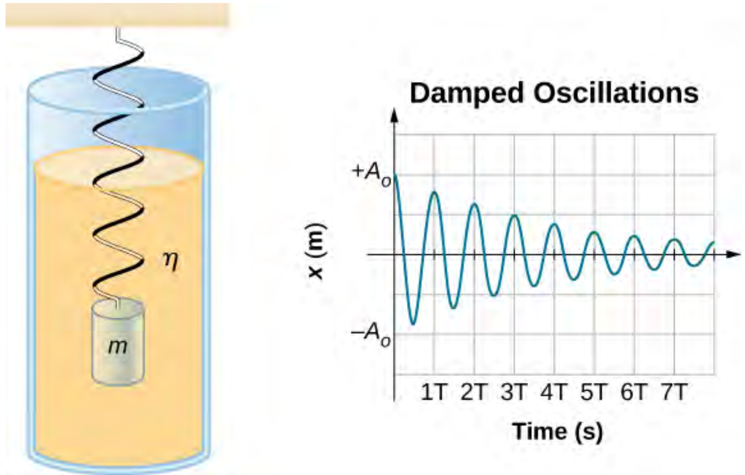


Figure 15.25 For a mass on a spring oscillating in a viscous fluid, the period remains constant, but the amplitudes of the oscillations decrease due to the damping caused by the fluid.

Consider the forces acting on the mass. Note that the only contribution of the weight is to change the equilibrium position, as discussed earlier in the chapter. Therefore, the net force is equal to the force of the spring and the damping force (F_D). If the magnitude of the velocity is small, meaning the mass oscillates slowly, the damping force is proportional to the velocity and acts against the direction of motion $(F_D = -bv)$. The net force on the mass is therefore

$$ma = -bv - kx.$$

Writing this as a differential equation in *x*, we obtain

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \quad (15.23)$$

To determine the solution to this equation, consider the plot of position versus time shown in **Figure 15.26**. The curve resembles a cosine curve oscillating in the envelope of an exponential function $A_0 e^{-\alpha t}$ where $\alpha = \frac{b}{2m}$. The solution is

$$x(t) = A_0 e^{-\frac{b}{2m}t} \cos(\omega t + \phi). \quad (15.24)$$

It is left as an exercise to prove that this is, in fact, the solution. To prove that it is the right solution, take the first and second derivatives with respect to time and substitute them into **Equation 15.23**. It is found that **Equation 15.24** is the solution if

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

Recall that the angular frequency of a mass undergoing SHM is equal to the square root of the force constant divided by the mass. This is often referred to as the **natural angular frequency**, which is represented as

$$\omega_0 = \sqrt{\frac{k}{m}}. \quad (15.25)$$

The angular frequency for damped harmonic motion becomes

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}. \quad (15.26)$$

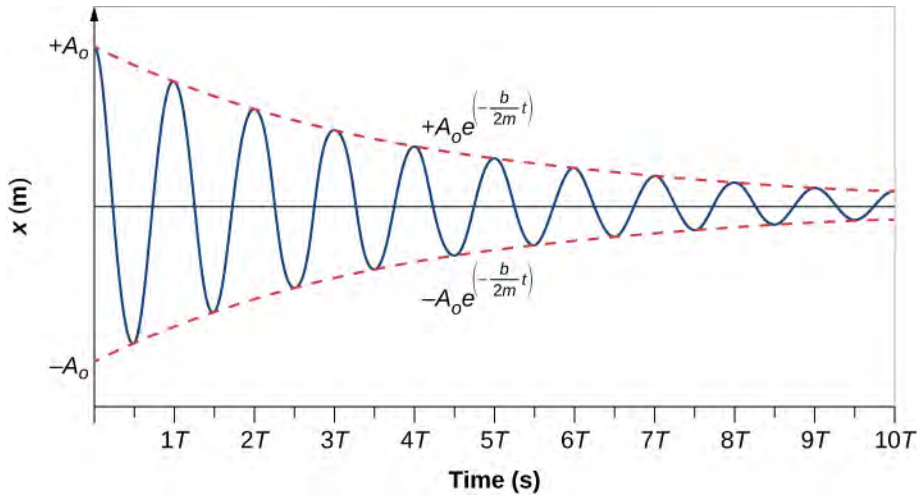


Figure 15.26 Position versus time for the mass oscillating on a spring in a viscous fluid. Notice that the curve appears to be a cosine function inside an exponential envelope.

Recall that when we began this description of damped harmonic motion, we stated that the damping must be small. Two questions come to mind. Why must the damping be small? And how small is small? If you gradually *increase* the amount of damping in a system, the period and frequency begin to be affected, because damping opposes and hence slows the back and forth motion. (The net force is smaller in both directions.) If there is very large damping, the system does not even oscillate—it slowly moves toward equilibrium. The angular frequency is equal to

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

As b increases, $\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ becomes smaller and eventually reaches zero when $b = \sqrt{4mk}$. If b becomes any larger, $\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ becomes a negative number and $\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$ is a complex number.

Figure 15.27 shows the displacement of a harmonic oscillator for different amounts of damping. When the damping constant is small, $b < \sqrt{4mk}$, the system oscillates while the amplitude of the motion decays exponentially. This system is said to be **underdamped**, as in curve (a). Many systems are underdamped, and oscillate while the amplitude decreases exponentially, such as the mass oscillating on a spring. The damping may be quite small, but eventually the mass comes to rest. If the damping constant is $b = \sqrt{4mk}$, the system is said to be **critically damped**, as in curve (b). An example of a critically damped system is the shock absorbers in a car. It is advantageous to have the oscillations decay as fast as possible. Here, the system does not oscillate, but asymptotically approaches the equilibrium condition as quickly as possible. Curve (c) in **Figure 15.27** represents an **overdamped** system where $b > \sqrt{4mk}$. An overdamped system will approach equilibrium over a longer period of time.

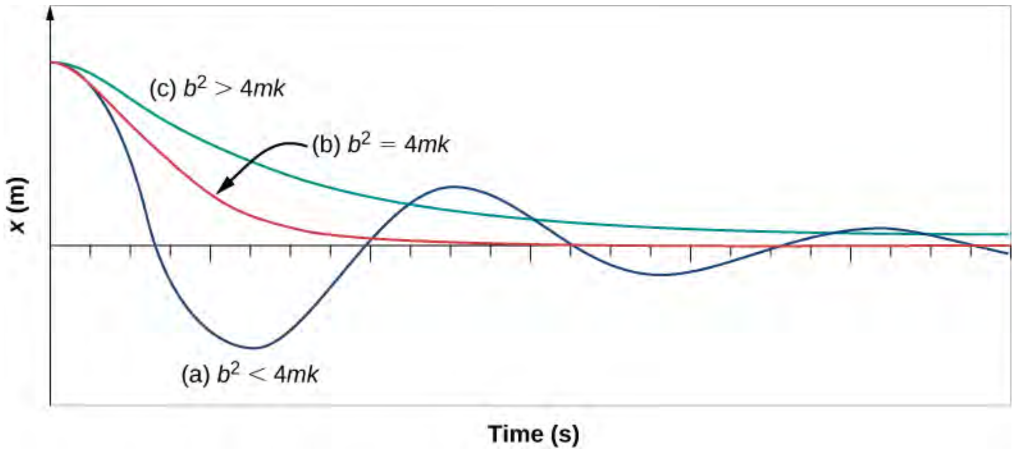


Figure 15.27 The position versus time for three systems consisting of a mass and a spring in a viscous fluid. (a) If the damping is small $\left(b < \sqrt{4mk}\right)$, the mass oscillates, slowly losing amplitude as the energy is dissipated by the non-conservative force(s). The limiting case is (b) where the damping is $\left(b = \sqrt{4mk}\right)$. (c) If the damping is very large $\left(b > \sqrt{4mk}\right)$, the mass does not oscillate when displaced, but attempts to return to the equilibrium position.

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position.

15.5 Check Your Understanding Why are completely undamped harmonic oscillators so rare?

15.6 | Forced Oscillations

Learning Objectives

By the end of this section, you will be able to:

- Define forced oscillations
- List the equations of motion associated with forced oscillations
- Explain the concept of resonance and its impact on the amplitude of an oscillator
- List the characteristics of a system oscillating in resonance

Sit in front of a piano sometime and sing a loud brief note at it with the dampers off its strings (**Figure 15.28**). It will sing the same note back at you—the strings, having the same frequencies as your voice, are resonating in response to the forces from the sound waves that you sent to them. This is a good example of the fact that objects—in this case, piano strings—can be forced to oscillate, and oscillate most easily at their natural frequency. In this section, we briefly explore applying a periodic driving force acting on a simple harmonic oscillator. The driving force puts energy into the system at a certain frequency, not necessarily the same as the natural frequency of the system. Recall that the natural frequency is the frequency at which a system would oscillate if there were no driving and no damping force.

Figure 15.28 You can cause the strings in a piano to vibrate simply by producing sound waves from your voice.

Most of us have played with toys involving an object supported on an elastic band, something like the paddle ball suspended from a finger in **Figure 15.29**. Imagine the finger in the figure is your finger. At first, you hold your finger steady, and the ball bounces up and down with a small amount of damping. If you move your finger up and down slowly, the ball follows along without bouncing much on its own. As you increase the frequency at which you move your finger up and down, the ball responds by oscillating with increasing amplitude. When you drive the ball at its natural frequency, the ball's oscillations increase in amplitude with each oscillation for as long as you drive it. The phenomenon of driving a system with a frequency equal to its natural frequency is called **resonance**. A system being driven at its natural frequency is said to *resonate*. As the driving frequency gets progressively higher than the resonant or natural frequency, the amplitude of the oscillations becomes smaller until the oscillations nearly disappear, and your finger simply moves up and down with little effect on the ball.

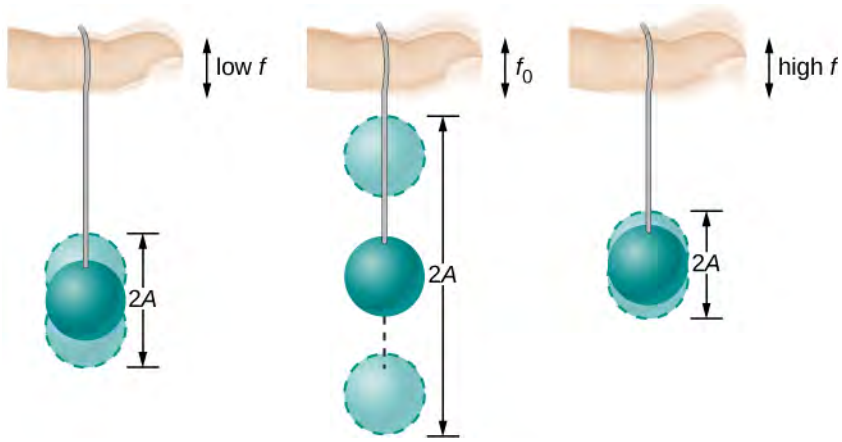


Figure 15.29 The paddle ball on its rubber band moves in response to the finger supporting it. If the finger moves with the natural frequency f_0 of the ball on the rubber band, then a resonance is achieved, and the amplitude of the ball's oscillations increases dramatically. At higher and lower driving frequencies, energy is transferred to the ball less efficiently, and it responds with lower-amplitude oscillations.

Consider a simple experiment. Attach a mass *m* to a spring in a viscous fluid, similar to the apparatus discussed in the damped harmonic oscillator. This time, instead of fixing the free end of the spring, attach the free end to a disk that is driven by a variable-speed motor. The motor turns with an angular driving frequency of ω. The rotating disk provides energy to the system by the work done by the driving force $\left(F_d = F_0 \sin(\omega t)\right)$. The experimental apparatus is shown in **Figure 15.30**.

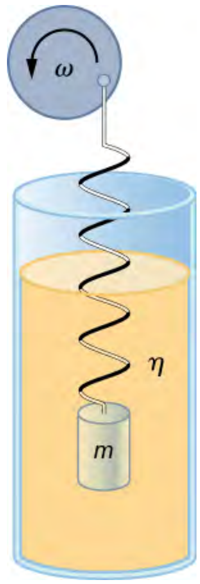


Figure 15.30 Forced, damped harmonic motion produced by driving a spring and mass with a disk driven by a variable-speed motor.

Using Newton's second law $(\vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}})$, we can analyze the motion of the mass. The resulting equation is similar to the force equation for the damped harmonic oscillator, with the addition of the driving force:

$$-kx - b\frac{dx}{dt} + F_0 \sin(\omega t) = m\frac{d^2x}{dt^2}. \quad (15.27)$$

When an oscillator is forced with a periodic driving force, the motion may seem chaotic. The motions of the oscillator is known as transients. After the transients die out, the oscillator reaches a steady state, where the motion is periodic. After some time, the steady state solution to this differential equation is

$$x(t) = A\cos(\omega t + \phi). \quad (15.28)$$

Once again, it is left as an exercise to prove that this equation is a solution. Taking the first and second time derivative of $x(t)$ and substituting them into the force equation shows that $x(t) = A\sin(\omega t + \phi)$ is a solution as long as the amplitude is equal to

$$A = \frac{F_0}{\sqrt{m\left(\omega^2 - \omega_0^2\right)^2 + b^2\omega^2}} \quad (15.29)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the system of the mass and spring. Recall that the angular frequency, and therefore the frequency, of the motor can be adjusted. Looking at the denominator of the equation for the amplitude, when the driving frequency is much smaller, or much larger, than the natural frequency, the square of the difference of the two angular frequencies $\left(\omega^2 - \omega_0^2\right)^2$ is positive and large, making the denominator large, and the result is a small amplitude for the oscillations of the mass. As the frequency of the driving force approaches the natural frequency of the system, the denominator becomes small and the amplitude of the oscillations becomes large. The maximum amplitude results when the frequency of the driving force equals the natural frequency of the system $\left(A_{\text{max}} = \frac{F_0}{b\omega}\right)$.

Figure 15.31 shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. Each of the three curves on the graph represents a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less energy is removed by the damping force. Note that since the amplitude grows as the damping decreases, taking this to the limit where there is no damping $(b = 0)$, the amplitude becomes infinite.

Note that a small-amplitude driving force can produce a large-amplitude response. This phenomenon is known as resonance. A common example of resonance is a parent pushing a small child on a swing. When the child wants to go higher, the parent does not move back and then, getting a running start, slam into the child, applying a great force in a short interval. Instead, the parent applies small pushes to the child at just the right frequency, and the amplitude of the child's swings increases.

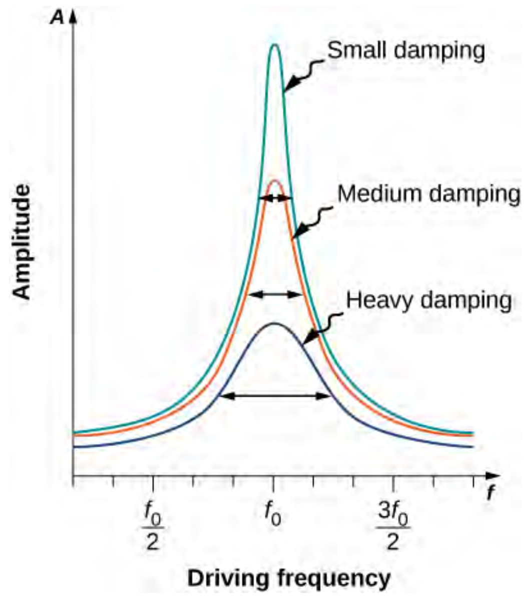


Figure 15.31 Amplitude of a harmonic oscillator as a function of the frequency of the driving force. The curves represent the same oscillator with the same natural frequency but with different amounts of damping. Resonance occurs when the driving frequency equals the natural frequency, and the greatest response is for the least amount of damping. The narrowest response is also for the least damping.

It is interesting to note that the widths of the resonance curves shown in Figure 15.31 depend on damping: the less the damping, the narrower the resonance. The consequence is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible. For instance, a radio has a circuit that is used to choose a particular radio station. In this case, the forced damped oscillator consists of a resistor, capacitor, and inductor, which will be discussed later in this course. The circuit is "tuned" to pick a particular radio station. Here it is desirable to have the resonance curve be very narrow, to pick out the exact frequency of the radio station chosen. The narrowness of the graph, and the ability to pick out a certain frequency, is known as the quality of the system. The quality is defined as the spread of the angular frequency, or equivalently, the spread in the frequency, at half the maximum amplitude, divided by the natural frequency $\left(Q = \frac{\Delta\omega}{\omega_0}\right)$ as shown in Figure 15.32. For a small damping, the quality is approximately equal to $Q \approx \frac{2b}{m}$.

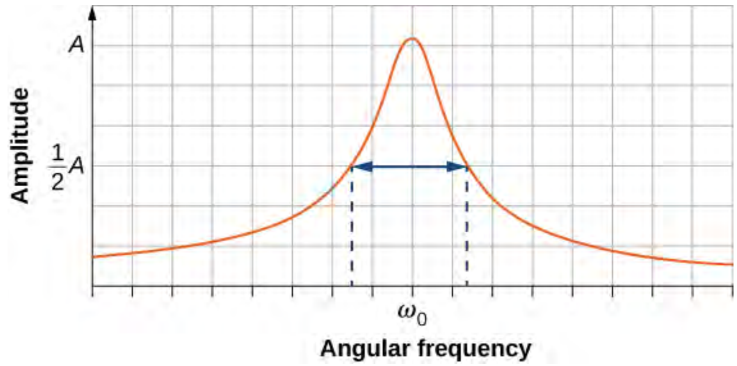


Figure 15.32 The quality of a system is defined as the spread in the frequencies at half the amplitude divided by the natural frequency.

These features of driven harmonic oscillators apply to a huge variety of systems. For instance, magnetic resonance imaging (MRI) is a widely used medical diagnostic tool in which atomic nuclei (mostly hydrogen nuclei or protons) are made to resonate by incoming radio waves (on the order of 100 MHz). In all of these cases, the efficiency of energy transfer from the driving force into the oscillator is best at resonance. **Figure 15.33** shows a photograph of a famous example (the Tacoma Narrows bridge) of the destructive effects of a driven harmonic oscillation. The Millennium bridge in London was closed for a short period of time for the same reason while inspections were carried out. Observations lead to modifications being made to the bridge prior to the reopening.

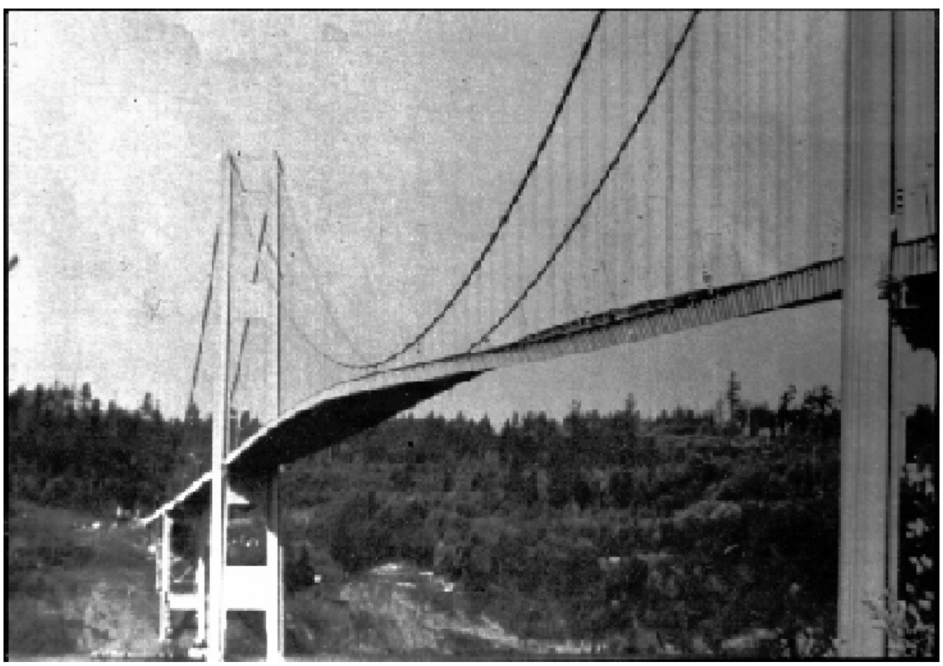


Figure 15.33 In 1940, the Tacoma Narrows bridge in the state of Washington collapsed. Moderately high, variable cross-winds (much slower than hurricane force winds) drove the bridge into oscillations at its resonant frequency. Damping decreased when support cables broke loose and started to slip over the towers, allowing increasingly greater amplitudes until the structure failed.

15.6 **Check Your Understanding** A famous magic trick involves a performer singing a note toward a crystal glass until the glass shatters. Explain why the trick works in terms of resonance and natural frequency.

CHAPTER 15 REVIEW

KEY TERMS

amplitude (*A*) maximum displacement from the equilibrium position of an object oscillating around the equilibrium position

critically damped condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

elastic potential energy potential energy stored as a result of deformation of an elastic object, such as the stretching of a spring

equilibrium position position where the spring is neither stretched nor compressed

force constant (*k*) characteristic of a spring which is defined as the ratio of the force applied to the spring to the displacement caused by the force

frequency (*f*) number of events per unit of time

natural angular frequency angular frequency of a system oscillating in SHM

oscillation single fluctuation of a quantity, or repeated and regular fluctuations of a quantity, between two extreme values around an equilibrium or average value

overdamped condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system

period (*T*) time taken to complete one oscillation

periodic motion motion that repeats itself at regular time intervals

phase shift angle, in radians, that is used in a cosine or sine function to shift the function left or right, used to match up the function with the initial conditions of data

physical pendulum any extended object that swings like a pendulum

resonance large amplitude oscillations in a system produced by a small amplitude driving force, which has a frequency equal to the natural frequency

restoring force force acting in opposition to the force caused by a deformation

simple harmonic motion (SHM) oscillatory motion in a system where the restoring force is proportional to the displacement, which acts in the direction opposite to the displacement

simple harmonic oscillator a device that oscillates in SHM where the restoring force is proportional to the displacement and acts in the direction opposite to the displacement

simple pendulum point mass, called a pendulum bob, attached to a near massless string

stable equilibrium point point where the net force on a system is zero, but a small displacement of the mass will cause a restoring force that points toward the equilibrium point

torsional pendulum any suspended object that oscillates by twisting its suspension

underdamped condition in which damping of an oscillator causes the amplitude of oscillations of a damped harmonic oscillator to decrease over time, eventually approaching zero

KEY EQUATIONS

Relationship between frequency and period	$f = \frac{1}{T}$
Position in SHM with $\phi = 0.00$	$x(t) = A\cos(\omega t)$
General position in SHM	$x(t) = A\cos(\omega t + \phi)$

General velocity in SHM	$v(t) = -A\omega \sin(\omega t + \phi)$
General acceleration in SHM	$a(t) = -A\omega^2 \cos(\omega t + \phi)$
Maximum displacement (amplitude) of SHM	$x_{\max} = A$
Maximum velocity of SHM	$\lvert v_{\max} \rvert = A\omega$
Maximum acceleration of SHM	$\lvert a_{\max} \rvert = A\omega^2$
Angular frequency of a mass-spring system in SHM	$\omega = \sqrt{\frac{k}{m}}$
Period of a mass-spring system in SHM	$T = 2\pi\sqrt{\frac{m}{k}}$
Frequency of a mass-spring system in SHM	$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
Energy in a mass-spring system in SHM	$E_{\text{Total}} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$
The velocity of the mass in a spring-mass system in SHM	$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$
The *x*-component of the radius of a rotating disk	$x(t) = A\cos(\omega t + \phi)$
The *x*-component of the velocity of the edge of a rotating disk	$v(t) = -v_{\max}\sin(\omega t + \phi)$
The *x*-component of the acceleration of the edge of a rotating disk	$a(t) = -a_{\max}\cos(\omega t + \phi)$
Force equation for a simple pendulum	$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$
Angular frequency for a simple pendulum	$\omega = \sqrt{\frac{g}{L}}$
Period of a simple pendulum	$T = 2\pi\sqrt{\frac{L}{g}}$
Angular frequency of a physical pendulum	$\omega = \sqrt{\frac{mgL}{I}}$
Period of a physical pendulum	$T = 2\pi\sqrt{\frac{I}{mgL}}$
Period of a torsional pendulum	$T = 2\pi\sqrt{\frac{I}{\kappa}}$
Newton’s second law for harmonic motion	$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$
Solution for underdamped harmonic motion	$x(t) = A_0 e^{-\frac{b}{2m}t}\cos(\omega t + \phi)$
Natural angular frequency of a mass-spring system	$\omega_0 = \sqrt{\frac{k}{m}}$
Angular frequency of underdamped harmonic motion	$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

Newton's second law for forced, damped oscillation	$-kx - b\frac{dx}{dt} + F_o \sin(\omega t) = m\frac{d^2x}{dt^2}$
Solution to Newton's second law for forced, damped oscillations	$x(t) = A\cos(\omega t + \phi)$
Amplitude of system undergoing forced, damped oscillations	$A = \frac{F_o}{\sqrt{m(\omega^2 - \omega_o^2)^2 + b^2\omega^2}}$

SUMMARY

15.1 Simple Harmonic Motion

- Periodic motion is a repeating oscillation. The time for one oscillation is the period T and the number of oscillations per unit time is the frequency f. These quantities are related by $f = \frac{1}{T}$.
- Simple harmonic motion (SHM) is oscillatory motion for a system where the restoring force is proportional to the displacement and acts in the direction opposite to the displacement.
- Maximum displacement is the amplitude A. The angular frequency ω, period T, and frequency f of a simple harmonic oscillator are given by $\omega = \sqrt{\frac{k}{m}}$, $T = 2\pi\sqrt{\frac{m}{k}}$, and $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where m is the mass of the system and k is the force constant.
- Displacement as a function of time in SHM is given by $x(t) = A\cos\left(\frac{2\pi}{T}t + \phi\right) = A\cos(\omega t + \phi)$.
- The velocity is given by $v(t) = -A\omega\sin(\omega t + \phi) = -v_{max}\sin(\omega t + \phi)$, where $v_{max} = A\omega = A\sqrt{\frac{k}{m}}$.
- The acceleration is $a(t) = -A\omega^2\cos(\omega t + \phi) = -a_{max}\cos(\omega t + \phi)$, where $a_{max} = A\omega^2 = A\frac{k}{m}$.

15.2 Energy in Simple Harmonic Motion

- The simplest type of oscillations are related to systems that can be described by Hooke's law, $F = -kx$, where F is the restoring force, x is the displacement from equilibrium or deformation, and k is the force constant of the system.
- Elastic potential energy U stored in the deformation of a system that can be described by Hooke's law is given by $U = \frac{1}{2}kx^2$.
- Energy in the simple harmonic oscillator is shared between elastic potential energy and kinetic energy, with the total being constant:

$$E_{Total} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant.}$$

- The magnitude of the velocity as a function of position for the simple harmonic oscillator can be found by using

$$|v| = \sqrt{\frac{k}{m}(A^2 - x^2)}.$$

15.3 Comparing Simple Harmonic Motion and Circular Motion

- A projection of uniform circular motion undergoes simple harmonic oscillation.
- Consider a circle with a radius A, moving at a constant angular speed ω. A point on the edge of the circle moves at a constant tangential speed of $v_{max} = A\omega$. The projection of the radius onto the x-axis is $x(t) = A\cos(\omega t + \phi)$, where (ϕ) is the phase shift. The x-component of the tangential velocity is $v(t) = -A\omega\sin(\omega t + \phi)$.

15.4 Pendulums

- A mass m suspended by a wire of length L and negligible mass is a simple pendulum and undergoes SHM for amplitudes less than about $15°$. The period of a simple pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the string and g is the acceleration due to gravity.
- The period of a physical pendulum $T = 2\pi\sqrt{\frac{I}{mgL}}$ can be found if the moment of inertia is known. The length between the point of rotation and the center of mass is L.
- The period of a torsional pendulum $T = 2\pi\sqrt{\frac{I}{\kappa}}$ can be found if the moment of inertia and torsion constant are known.

15.5 Damped Oscillations

- Damped harmonic oscillators have non-conservative forces that dissipate their energy.
- Critical damping returns the system to equilibrium as fast as possible without overshooting.
- An underdamped system will oscillate through the equilibrium position.
- An overdamped system moves more slowly toward equilibrium than one that is critically damped.

15.6 Forced Oscillations

- A system's natural frequency is the frequency at which the system oscillates if not affected by driving or damping forces.
- A periodic force driving a harmonic oscillator at its natural frequency produces resonance. The system is said to resonate.
- The less damping a system has, the higher the amplitude of the forced oscillations near resonance. The more damping a system has, the broader response it has to varying driving frequencies.

CONCEPTUAL QUESTIONS

15.1 Simple Harmonic Motion

1. What conditions must be met to produce SHM?

2. (a) If frequency is not constant for some oscillation, can the oscillation be SHM? (b) Can you think of any examples of harmonic motion where the frequency may depend on the amplitude?

3. Give an example of a simple harmonic oscillator, specifically noting how its frequency is independent of amplitude.

4. Explain why you expect an object made of a stiff material to vibrate at a higher frequency than a similar object made of a more pliable material.

5. As you pass a freight truck with a trailer on a highway, you notice that its trailer is bouncing up and down slowly. Is it more likely that the trailer is heavily loaded or nearly empty? Explain your answer.

6. Some people modify cars to be much closer to the ground than when manufactured. Should they install stiffer springs? Explain your answer.

15.2 Energy in Simple Harmonic Motion

7. Describe a system in which elastic potential energy is stored.

8. Explain in terms of energy how dissipative forces such as friction reduce the amplitude of a harmonic oscillator. Also explain how a driving mechanism can compensate. (A pendulum clock is such a system.)

9. The temperature of the atmosphere oscillates from a maximum near noontime and a minimum near sunrise. Would you consider the atmosphere to be in stable or unstable equilibrium?

15.3 Comparing Simple Harmonic Motion and Circular Motion

10. Can this analogy of SHM to circular motion be carried out with an object oscillating on a spring vertically hung from the ceiling? Why or why not? If given the choice, would you prefer to use a sine function or a cosine function to model the motion?

11. If the maximum speed of the mass attached to a spring, oscillating on a frictionless table, was increased, what characteristics of the rotating disk would need to be changed?

15.4 Pendulums

12. Pendulum clocks are made to run at the correct rate by adjusting the pendulum's length. Suppose you move from one city to another where the acceleration due to gravity is slightly greater, taking your pendulum clock with you, will you have to lengthen or shorten the pendulum to keep the correct time, other factors remaining constant? Explain your answer.

13. A pendulum clock works by measuring the period of a pendulum. In the springtime the clock runs with perfect time, but in the summer and winter the length of the pendulum changes. When most materials are heated, they expand. Does the clock run too fast or too slow in the summer? What about the winter?

14. With the use of a phase shift, the position of an object may be modeled as a cosine or sine function. If given the option, which function would you choose? Assuming that the phase shift is zero, what are the initial conditions of function; that is, the initial position, velocity, and acceleration, when using a sine function? How about when a cosine function is used?

15.5 Damped Oscillations

15. Give an example of a damped harmonic oscillator. (They are more common than undamped or simple harmonic oscillators.)

16. How would a car bounce after a bump under each of these conditions?

(a) overdamping

(b) underdamping

(c) critical damping

17. Most harmonic oscillators are damped and, if undriven, eventually come to a stop. Why?

15.6 Forced Oscillations

18. Why are soldiers in general ordered to "route step" (walk out of step) across a bridge?

19. Do you think there is any harmonic motion in the physical world that is not damped harmonic motion? Try to make a list of five examples of undamped harmonic motion and damped harmonic motion. Which list was easier to make?

20. Some engineers use sound to diagnose performance problems with car engines. Occasionally, a part of the engine is designed that resonates at the frequency of the engine. The unwanted oscillations can cause noise that irritates the driver or could lead to the part failing prematurely. In one case, a part was located that had a length *L* made of a material with a mass *M*. What can be done to correct this problem?

PROBLEMS

15.1 Simple Harmonic Motion

21. Prove that using $x(t) = A\sin(\omega t + \phi)$ will produce the same results for the period for the oscillations of a mass and a spring. Why do you think the cosine function was chosen?

22. What is the period of 60.0 Hz of electrical power?

23. If your heart rate is 150 beats per minute during strenuous exercise, what is the time per beat in units of seconds?

24. Find the frequency of a tuning fork that takes $2.50 \times 10^{-3}\,\text{s}$ to complete one oscillation.

25. A stroboscope is set to flash every $8.00 \times 10^{-5}\,\text{s}$. What is the frequency of the flashes?

26. A tire has a tread pattern with a crevice every 2.00 cm. Each crevice makes a single vibration as the tire moves. What is the frequency of these vibrations if the car moves at 30.0 m/s?

27. Each piston of an engine makes a sharp sound every other revolution of the engine. (a) How fast is a race car going if its eight-cylinder engine emits a sound of frequency 750 Hz, given that the engine makes 2000 revolutions per kilometer? (b) At how many revolutions per minute is the engine rotating?

28. A type of cuckoo clock keeps time by having a mass bouncing on a spring, usually something cute like a cherub in a chair. What force constant is needed to produce a period of 0.500 s for a 0.0150-kg mass?

29. A mass m_0 is attached to a spring and hung vertically. The mass is raised a short distance in the vertical direction and released. The mass oscillates with a frequency f_0. If the mass is replaced with a mass nine times as large, and the experiment was repeated, what would be the frequency of the oscillations in terms of f_0 ?

30. A 0.500-kg mass suspended from a spring oscillates with a period of 1.50 s. How much mass must be added to the object to change the period to 2.00 s?

31. By how much leeway (both percentage and mass) would you have in the selection of the mass of the object in the previous problem if you did not wish the new period to be greater than 2.01 s or less than 1.99 s?

15.2 Energy in Simple Harmonic Motion

32. Fish are hung on a spring scale to determine their mass. (a) What is the force constant of the spring in such a scale if it the spring stretches 8.00 cm for a 10.0 kg load? (b) What is the mass of a fish that stretches the spring 5.50 cm? (c) How far apart are the half-kilogram marks on the scale?

33. It is weigh-in time for the local under-85-kg rugby team. The bathroom scale used to assess eligibility can be described by Hooke's law and is depressed 0.75 cm by its maximum load of 120 kg. (a) What is the spring's effective force constant? (b) A player stands on the scales and depresses it by 0.48 cm. Is he eligible to play on this under-85-kg team?

34. One type of BB gun uses a spring-driven plunger to blow the BB from its barrel. (a) Calculate the force constant of its plunger's spring if you must compress it 0.150 m to drive the 0.0500-kg plunger to a top speed of 20.0 m/s. (b) What force must be exerted to compress the spring?

35. When an 80.0-kg man stands on a pogo stick, the spring is compressed 0.120 m. (a) What is the force constant of the spring? (b) Will the spring be compressed more when he hops down the road?

36. A spring has a length of 0.200 m when a 0.300-kg mass hangs from it, and a length of 0.750 m when a 1.95-kg mass hangs from it. (a) What is the force constant of the spring? (b) What is the unloaded length of the spring?

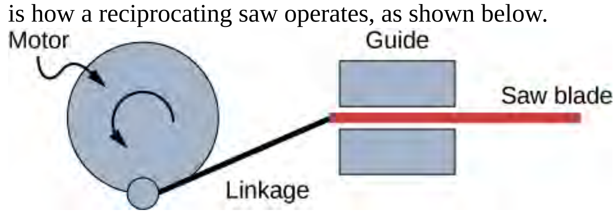
37. The length of nylon rope from which a mountain climber is suspended has an effective force constant of 1.40×10^4 N/m . (a) What is the frequency at which he bounces, given his mass plus and the mass of his equipment are 90.0 kg? (b) How much would this rope stretch to break the climber's fall if he free-falls 2.00 m before the rope runs out of slack? (*Hint:* Use conservation of energy.) (c) Repeat both parts of this problem in the situation where twice this length of nylon rope is used.

15.3 Comparing Simple Harmonic Motion and Circular Motion

38. The motion of a mass on a spring hung vertically, where the mass oscillates up and down, can also be modeled using the rotating disk. Instead of the lights being placed horizontally along the top and pointing down, place the lights vertically and have the lights shine on the side of the rotating disk. A shadow will be produced on a nearby wall, and will move up and down. Write the equations of motion for the shadow taking the position at $t = 0.0\,\text{s}$ to be $y = 0.0\,\text{m}$ with the mass moving in the positive *y*-direction.

39. (a) A novelty clock has a 0.0100-kg-mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position? (b) How many joules of kinetic energy does the object have at its maximum velocity?

40. Reciprocating motion uses the rotation of a motor to produce linear motion up and down or back and forth. This is how a reciprocating saw operates, as shown below.



If the motor rotates at 60 Hz and has a radius of 3.0 cm, estimate the maximum speed of the saw blade as it moves up and down. This design is known as a scotch yoke.

41. A student stands on the edge of a merry-go-round which rotates five times a minute and has a radius of two meters one evening as the sun is setting. The student produces a shadow on the nearby building. (a) Write an equation for the position of the shadow. (b) Write an equation for the velocity of the shadow.

15.4 Pendulums

42. What is the length of a pendulum that has a period of 0.500 s?

43. Some people think a pendulum with a period of 1.00 s can be driven with "mental energy" or psycho kinetically, because its period is the same as an average heartbeat. True or not, what is the length of such a pendulum?

44. What is the period of a 1.00-m-long pendulum?

45. How long does it take a child on a swing to complete one swing if her center of gravity is 4.00 m below the pivot?

46. The pendulum on a cuckoo clock is 5.00-cm long. What is its frequency?

47. Two parakeets sit on a swing with their combined CMs 10.0 cm below the pivot. At what frequency do they swing?

48. (a) A pendulum that has a period of 3.00000 s and that is located where the acceleration due to gravity is 9.79 m/s^2 is moved to a location where the acceleration due to gravity is 9.82 m/s^2. What is its new period? (b) Explain why so many digits are needed in the value for the period, based on the relation between the period and the acceleration due to gravity.

49. A pendulum with a period of 2.00000 s in one location ($g = 9.80 \text{m/s}^2$) is moved to a new location where the period is now 1.99796 s. What is the acceleration due to gravity at its new location?

50. (a) What is the effect on the period of a pendulum if you double its length? (b) What is the effect on the period of a pendulum if you decrease its length by 5.00%?

15.5 Damped Oscillations

51. The amplitude of a lightly damped oscillator decreases by 3.0% during each cycle. What percentage of the mechanical energy of the oscillator is lost in each cycle?

15.6 Forced Oscillations

52. How much energy must the shock absorbers of a 1200-kg car dissipate in order to damp a bounce that initially has a velocity of 0.800 m/s at the equilibrium position? Assume the car returns to its original vertical position.

53. If a car has a suspension system with a force constant of $5.00 \times 10^4 \text{ N/m}$, how much energy must the car's shocks remove to dampen an oscillation starting with a maximum displacement of 0.0750 m?

54. (a) How much will a spring that has a force constant of 40.0 N/m be stretched by an object with a mass of 0.500 kg when hung motionless from the spring? (b) Calculate the decrease in gravitational potential energy of the 0.500-kg object when it descends this distance. (c) Part of this gravitational energy goes into the spring. Calculate the energy stored in the spring by this stretch, and compare it with the gravitational potential energy. Explain where the rest of the energy might go.

55. Suppose you have a 0.750-kg object on a horizontal surface connected to a spring that has a force constant of 150 N/m. There is simple friction between the object and surface with a static coefficient of friction $\mu_\text{s} = 0.100$. (a) How far can the spring be stretched without moving the mass? (b) If the object is set into oscillation with an amplitude twice the distance found in part (a), and the kinetic coefficient of friction is $\mu_\text{k} = 0.0850$, what total distance does it travel before stopping? Assume it starts at the maximum amplitude.

ADDITIONAL PROBLEMS

56. Suppose you attach an object with mass *m* to a vertical spring originally at rest, and let it bounce up and down. You release the object from rest at the spring's original rest length, the length of the spring in equilibrium, without the mass attached. The amplitude of the motion is the distance between the equilibrium position of the spring without the mass attached and the equilibrium position of the spring with the mass attached. (a) Show that the spring exerts an upward force of 2.00*mg* on the object at its lowest point. (b) If the spring has a force constant of 10.0 N/m, is hung horizontally, and the position of the free end of the spring is marked as $y = 0.00\,\text{m}$, where is the new equilibrium position if a 0.25-kg-mass object is hung from the spring? (c) If the spring has a force constant of 10.0 M/m and a 0.25-kg-mass object is set in motion as described, find the amplitude of the oscillations. (d) Find the maximum velocity.

57. A diver on a diving board is undergoing SHM. Her mass is 55.0 kg and the period of her motion is 0.800 s. The next diver is a male whose period of simple harmonic oscillation is 1.05 s. What is his mass if the mass of the board is negligible?

58. Suppose a diving board with no one on it bounces up and down in a SHM with a frequency of 4.00 Hz. The board has an effective mass of 10.0 kg. What is the frequency of the SHM of a 75.0-kg diver on the board?

59. The device pictured in the following figure entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring. (a) If the spring stretches 0.250 m while supporting an 8.0-kg child, what is its force constant? (b) What is the time for one complete bounce of this child? (c) What is the child's maximum velocity if the amplitude of her bounce is 0.200 m?

Figure 15.34 (credit: Lisa Doehnert)

60. A mass is placed on a frictionless, horizontal table. A spring $(k = 100\,\text{N/m})$, which can be stretched or compressed, is placed on the table. A 5.00-kg mass is attached to one end of the spring, the other end is anchored to the wall. The equilibrium position is marked at zero. A student moves the mass out to $x = 4.0\text{cm}$ and releases it from rest. The mass oscillates in SHM. (a) Determine the equations of motion. (b) Find the position, velocity, and acceleration of the mass at time $t = 3.00\,\text{s}$.

61. Find the ratio of the new/old periods of a pendulum if the pendulum were transported from Earth to the Moon, where the acceleration due to gravity is $1.63\,\text{m/s}^2$.

62. At what rate will a pendulum clock run on the Moon, where the acceleration due to gravity is $1.63\,\text{m/s}^2$, if it keeps time accurately on Earth? That is, find the time (in hours) it takes the clock's hour hand to make one revolution on the Moon.

63. If a pendulum-driven clock gains 5.00 s/day, what fractional change in pendulum length must be made for it to keep perfect time?

64. A 2.00-kg object hangs, at rest, on a 1.00-m-long string attached to the ceiling. A 100-g mass is fired with a speed of 20 m/s at the 2.00-kg mass, and the 100.00-g mass collides perfectly elastically with the 2.00-kg mass. Write an equation for the motion of the hanging mass after the collision. Assume air resistance is negligible.

65. A 2.00-kg object hangs, at rest, on a 1.00-m-long string attached to the ceiling. A 100-g object is fired with a speed of 20 m/s at the 2.00-kg object, and the two objects collide and stick together in a totally inelastic collision. Write an equation for the motion of the system after the collision. Assume air resistance is negligible.

66. Assume that a pendulum used to drive a grandfather clock has a length $L_0 = 1.00\,\text{m}$ and a mass *M* at temperature $T = 20.00°\text{C}$. It can be modeled as a physical pendulum as a rod oscillating around one end. By what percentage will the period change if the temperature increases by $10°\text{C}$? Assume the length of the rod changes linearly with temperature, where $L = L_0(1 + \alpha\Delta T)$ and the rod is made of brass $\left(\alpha = 18 \times 10^{-6}\,°\text{C}^{-1}\right)$.

67. A 2.00-kg block lies at rest on a frictionless table. A spring, with a spring constant of 100 N/m is attached to the wall and to the block. A second block of 0.50 kg is placed on top of the first block. The 2.00-kg block is gently pulled to a position $x = +A$ and released from rest. There is a coefficient of friction of 0.45 between the two blocks. (a) What is the period of the oscillations? (b) What is the largest amplitude of motion that will allow the blocks to oscillate without the 0.50-kg block sliding off?

CHALLENGE PROBLEMS

68. A suspension bridge oscillates with an effective force constant of 1.00×10^8 N/m. (a) How much energy is needed to make it oscillate with an amplitude of 0.100 m? (b) If soldiers march across the bridge with a cadence equal to the bridge's natural frequency and impart 1.00×10^4 J of energy each second, how long does it take for the bridge's oscillations to go from 0.100 m to 0.500 m amplitude.

69. Near the top of the Citigroup Center building in New York City, there is an object with mass of 4.00×10^5 kg on springs that have adjustable force constants. Its function is to dampen wind-driven oscillations of the building by oscillating at the same frequency as the building is being driven—the driving force is transferred to the object, which oscillates instead of the entire building. (a) What effective force constant should the springs have to make the object oscillate with a period of 2.00 s? (b) What energy is stored in the springs for a 2.00-m displacement from equilibrium?

70. Parcels of air (small volumes of air) in a stable atmosphere (where the temperature increases with height) can oscillate up and down, due to the restoring force provided by the buoyancy of the air parcel. The frequency of the oscillations are a measure of the stability of the atmosphere. Assuming that the acceleration of an air parcel can be modeled as $\frac{\partial^2 z'}{\partial t^2} = \frac{g}{\rho_o} \frac{\partial \rho(z)}{\partial z} z'$, prove that $z' = z_0{}' e^{t\sqrt{-N^2}}$ is a solution, where N is known as the Brunt-Väisälä frequency. Note that in a stable atmosphere, the density decreases with height and parcel oscillates up and down.

71. Consider the van der Waals potential $U(r) = U_o\left[\left(\frac{R_o}{r}\right)^{12} - 2\left(\frac{R_o}{r}\right)^6\right]$, used to model the potential energy function of two molecules, where the minimum potential is at $r = R_o$. Find the force as a function of r. Consider a small displacement $r = R_o + r'$ and use the binomial theorem:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

,

to show that the force does approximate a Hooke's law force.

72. Suppose the length of a clock's pendulum is changed by 1.000%, exactly at noon one day. What time will the clock read 24.00 hours later, assuming it the pendulum has kept perfect time before the change? Note that there are two answers, and perform the calculation to four-digit precision.

73. (a) The springs of a pickup truck act like a single spring with a force constant of 1.30×10^5 N/m. By how much will the truck be depressed by its maximum load of 1000 kg? (b) If the pickup truck has four identical springs, what is the force constant of each?