# Explorations in Geometry

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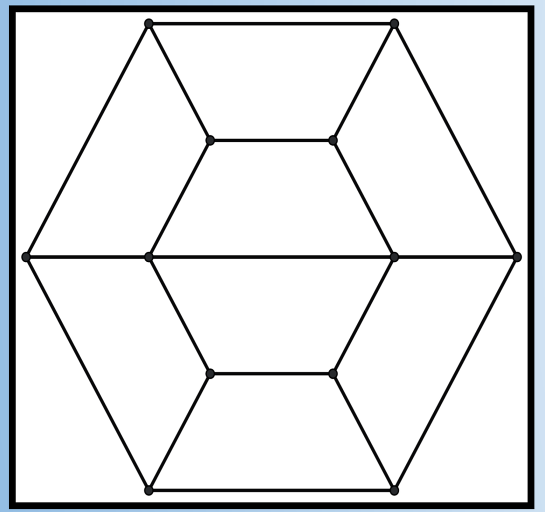
**Abstract:** Dynamic geometric software can facilitate mathematical conjecturing. Computer construction programs allow the users to construct geometric shapes with specific constraints. Once these geometric shapes are constructed, they can be moved around on the computer screen without changing the specified constraints. This dynamic aspect of the software allows the users to conjecture about the relationships between and among different properties of the created shapes. The purpose of this poster is to display examples of geometric conjectures that can be explored in a dynamic geometry environment. These conjectures can be presented as geometry problems. Some of these problems (when solved) have the potential to emerge as significant mathematical theorems.

**Problem 1:**

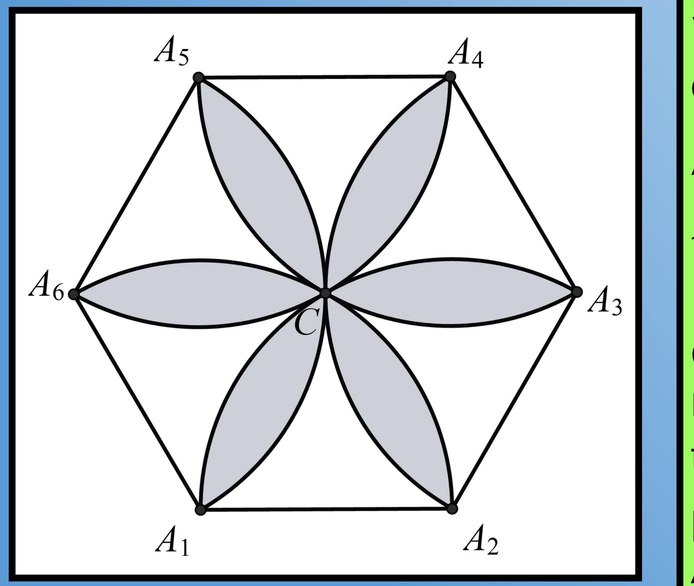
The figure below shows a regular hexagon divided into eight congruent trapezoids. If the length of each side of the regular hexagon is 1 unit, determine the lengths of each of the remaining three sides of each trapezoid.



We can show that each trapezoid can be divided further into three congruent equilateral triangles. The length of each side of each equilateral triangle will be ½. Therefore, we an conclude that the longest side of the trapezoid is 1, and the remaining three sides are each ½.

**Problem 2:**

Suppose regular hexagon $A$ has its vertices at points $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, and $A_6$. Six congruent circular arcs pass through point $C$, the center of hexagon $A$. These six circular arcs enclose six petal-shaped disjoint regions inside hexagon $A$. These six petal-shaped regions have been shaded. If $A_1A_2 = 1$, find the total area of the shaded petal-shaped regions. Your answer must be exact!




Suppose we have a regular hexagon with vertices and arcs as described and that $A_1A_2 = 1$. Since we have a regular hexagon, all sides have a length of 1 and the hexagon can be divided into six congruent equilateral triangles such that $\Delta A_1A_2C \cong \Delta A_2A_3C \cong \Delta A_3A_4C \cong \Delta A_4A_5C \cong \Delta A_5A_6C \cong \Delta A_6A_1C$. The area of an equilateral triangle with a side of length 1 is $\sqrt{3}/4$.

Each petal is part is formed with the intersection of arcs, each of which is a part of a unit circle. The area of a circle with radius 1 is $\pi$. Each angle in a regular hexagon has a measure of 120, thus each arc is 1/3 of a unit circle. It follows that the sector enclosed by that arc has an area of $\frac{\pi}{3}$. This sector equals two of the equilateral triangles plus two ½ petals. That is,

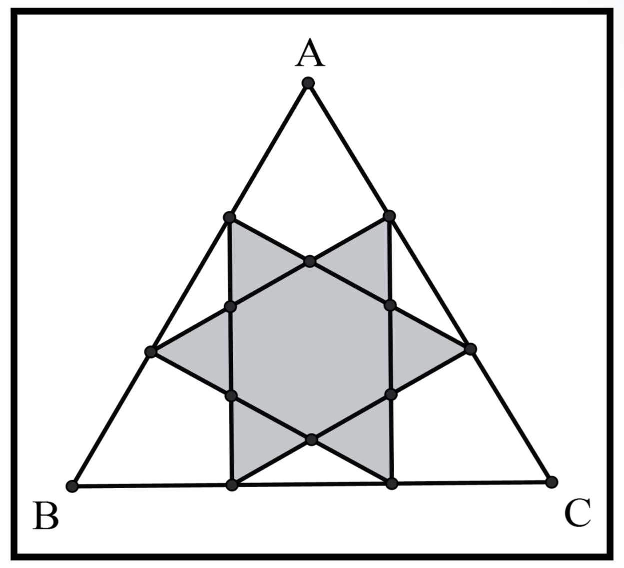
Area of sector = two equilateral triangles plus one petal.

$\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{4}\right)$ + area of one petal . Thus, area of one petal $= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$.

Since there are 6 petals, the area of the shaded region is $6\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = 2\pi - 3\sqrt{3}$.

**Problem 3:**

Triangle $ABC$ is equilateral. Two points split each side of $\Delta ABC$ into 3 equal parts. These points are connected to one another as shown. Line segments connecting these points also intersect with one another as shown. In the figure below a dodecagon (12-sided polygon) shaped like a star is shaded gray. If the area of $\Delta ABC$ is 9 sq. units, find the area of the gray dodecagon.



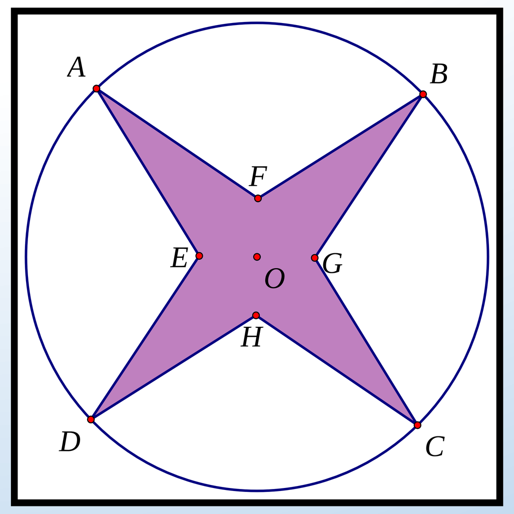

Let $BC = a$. Then the following is true: $a^2\frac{\sqrt{3}}{4} = 9$. Therefore, $a = \frac{6}{\sqrt{\sqrt{3}}}$.

Therefore, the longest side of each of the three white isosceles triangle must be $\frac{2}{\sqrt{\sqrt{3}}}$. We can also show that each white isosceles triangle is a 30º-30º-120º triangle. Therefore, each of the congruent sides of each white isosceles triangle must be $\frac{2}{\sqrt{3}\sqrt{\sqrt{3}}} = \frac{2}{3^{3/4}}$.

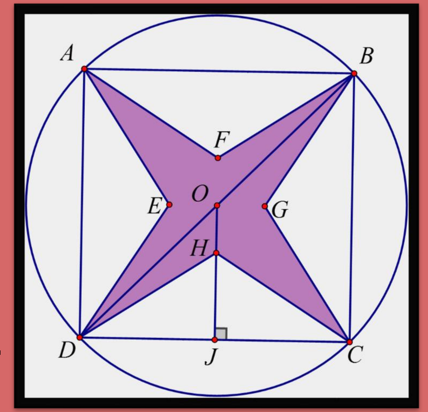
We can show that the shaded dodecagon is comprised of 12 equilateral triangles, and each side of each of these 12 equilateral triangle must be $\frac{2}{3^{3/4}}$. Therefore, the area of the shaded region must be be $12 \times \frac{\sqrt{3}}{4} \times \left(\frac{2}{3^{3/4}}\right)^2 = 4$

**Problem 4:**

Circle with center O has a radius of 3.5 cm. Points $A$, $B$, $C$, and $D$ are placed equidistant around the circumference of the circle. The length from $O$ to point $E$, $F$, $G$, and $H$ is one fourth the radius. Moreover, $\angle AFB = \angle BGC = \angle CHD = \angle DEA$. What is the area of the shaded star-shaped octagon?




The area of square $ABCD = \frac{7^2}{2} = 24.5$ cm². By applying the Pythagorean Theorem to right $\Delta ABD$ we get $AD = \frac{7}{\sqrt{2}}$ . Therefore, $JO = \frac{AD}{2} = \frac{7}{2\sqrt{2}}$. Since $OH = \frac{7}{8}$, $\frac{OH}{OJ} = \frac{\frac{7}{8}}{\frac{7}{2\sqrt{2}}} = \frac{\sqrt{2}}{4}$ .




Therefore, the area of $\Delta DOH$ : the area of $\Delta DOJ = \frac{\sqrt{2}}{4}$ .

Therefore, the area of octagon $AFBGCHDE$ : the area of square $ABCD = \frac{\sqrt{2}}{4}$ .

Since, the area of square $ABCD = \frac{49}{2}$ cm², the area of octagon $AFBGCHDE = \frac{\sqrt{2}}{4} * \frac{49}{2} = \frac{49\sqrt{2}}{8}$ cm².

**References:**

[1] M. DE VILLIERS, *Rethinking Proof with Geometer's Sketchpad*, Key Curriculum Press Berkeley, CA, 2003.